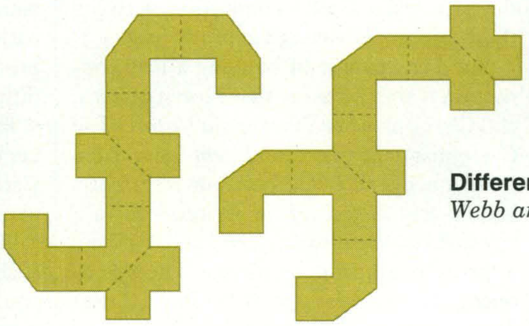
You Can't Hear the Shape of a Drum

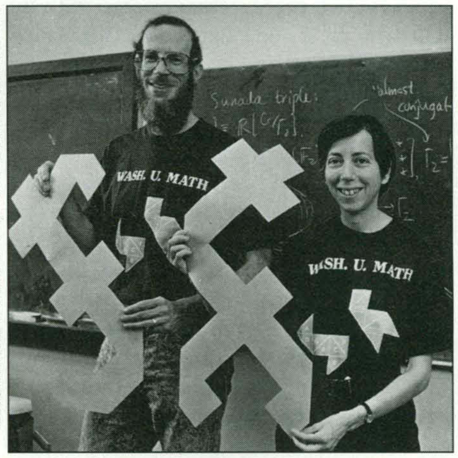
Marching to the beat of your own drummer just got a little harder, according to a new result in mathematics

THAT GREAT JAZZ SOLOIST "HANK" DAVID Thoreau used to urge each individual to follow the beat of his or her own drummer. But he never mentioned the nonconformist's nightmare: What if you couldn't tell the sound of your own drummer from someone else's?

A recent mathematical discovery suggests that's entirely possible. Carolyn Gordon and David Webb at Washington University in St. Louis and Scott Wolpert at the University of Maryland have found two different geometric shapes that produce the exact same mathematical "sounds." Made into real drums, with drumheads of the same material and tension, the two shapes would resonate at exactly the same frequencies. The discovery of these sound-alike drums caps 25 years of speculation and progress, which began in 1966 when Mark Kac of Rockefeller University asked the simple-sounding question, "Can you hear the shape of a drum?"

Kac wasn't trying to refine the design of kettle drums or bongos; instead he was posing a test of his colleagues' understanding of vibrating surfaces, and a challenge in pure mathematics. The kind of drum Kac had in mind, after all, is not exactly a standard musical instrument: It's nothing more than a two-dimensional shape in the plane. Kac's drum can be anything from the simplest circle to the most convoluted polygon, or even one of Mandelbrot's hideous fractals. All that's called for is an interior and a boundary. The interior does the vibrating; the boundary determines which frequencies are allowed.





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Different drums. *But by subdividing the shapes* (left), *Webb and Gordon proved they "sound" the same.*

Every drum, real or abstract, vibrates according to a mathematical formulation called the wave equation, which describes the physics not only of sound but also of other wave-like phenomena, including water waves and the wave functions of quantum mechanics. In the wave equation, it's the boundary conditions that determine the allowed frequencies. For example, a violin string—which Kac might call a one-dimensional drum—has to stay fixed at its endpoints; as a result, its frequencies are multiples of a single "fundamental" frequency that is determined, mathematically, by the length of the string. In the case of a drum, the equation's solutions—which describe the up-and-down motion of each point of the drum—have to leave the drum stationary at all points along its boundary. Otherwise, the drum would shake free of its frame.

Because the boundary of a drum has far more than two points, the set of frequencies at which it vibrates is, in general, far more complicated than a string's. However, one thing is certain: The boundary's shape determines the sound. If you build two drums with the exact same shape, they will make the exact same set of sounds.

You might think that statement works just as well in reverse: If you hear the exact same sounds coming from two drums, those drums necessarily have the same shape. After all, the sound of a drum clearly changes with its size or, more specifically, its area, as the difference between a snare and a bass drum illustrates. A drum's set of sounds also varies with the length of its perimeter. If the sound of a drum can tell that much about its shape, you might think the sound could reveal the shape itself.

By the time Kac posed his question, though, researchers already had a counterexample of sorts. In 1964, John Milnor, now at the State University of New York at Stony Brook, found a mismatched pair of 16-dimensional "toroidal drums" that made identical "sounds." These higher-dimensional drums are, of course, purely abstract, but the underlying mathematics is essentially the same as for more familiar drums. Since then, mathematicians found more sets of like-sounding drums in lower and lower dimensions.

Still, you definitely *can* hear the shape of a one-dimensional drum—that is, you can tell a string's length by its lowest sound. To find out which camp the two-dimensional drums belong to, Gordon, Webb, and Wolpert had to take a roundabout route. Fortunately, the way had been paved by plenty of other people. In the 1980s, Toshikazu Sunada of Nagoya University in Japan developed an algebraic framework that provided a new, systematic approach for comparing the sounds made by different shapes. Using Sunada's techniques, several mathematicians, including Robert Brooks at the University of Southern California and Peter Buser at the Ecole Polytechnique Federale in Lausanne, Switzerland, created sound-alike surfaces that no longer existed only in theory but could actually be constructed. Their examples, however, are three-dimensional rather than flat—more like bells than drums—so they didn't count as answers to Kac's question.

Not without modification, in any case. But when Gordon described one of Buser's examples at a geometry conference last spring, Wolpert, in the audience, noticed that a certain symmetry in Buser's example would allow it to be flattened in a natural way. He asked Gordon if the two-dimensional drums that would result from flattening two sound-alike bells might not be the long-sought answer to Kac's question.

Wolpert's question "was like a cold shower," recalls Webb. "It really made us sit up and think about this [example] again." Nevertheless, Gordon and Webb initially thought they'd have better luck with more complicated examples. "We convinced ourselves wrongly that this simple example wasn't going to work," Webb explains. So back in St. Louis, the two mathematicians, who are also husband and wife, spent days constructing complicated Buser-type bells out of paper, looking for good prospects for flattening. "We got these huge castles" made out of paper, Gordon recalls. "They took up our living room."

Finally, when their most promising example failed to flatten properly, Gordon and Webb returned to Buser's original example, which had been at the back of their minds all along. But when the pieces started falling into place, husband and wife were on opposite sides of the Atlantic: Webb at Dartmouth College, Gordon visiting Germany. They compensated with "a lot of transatlantic phone calls and twice-a-day faxes," Webb recalls.

By the time the couple got back together, in France, they had their first example of a pair of sound-alike drums. The two drums are based on bells formed of seven square crosses (like the Red Cross symbol, or the symbol on the Swiss flag), stitched together like the patches in a quilt. By folding each bell along several diagonals, Gordon and Webb ended up with two flat surfaces, each consisting of seven half-crosses (see figure).

To prove these drums were the answer, Gordon and Webb didn't take the obvious route of mathematically "beating" both drums to show they make the same sound—since such calculations can't be exact, a very small frequency difference could have escaped detection. Instead, their proof builds on work by Pierre Bérard of the University of Grenoble, who generalized Sunada's approach and showed how to take a solution of the wave equation on one drum and "transplant" it to the other. In essence, the proof takes a snapshot of each resonance on one drum, cuts it into seven pieces, one for each half-cross, and then reassembles these pieces into a picture of a standing wave on the other drum. If the reassembled picture looks smooth across the cuts and has the right behavior on the boundary, then the exact same solution works for the second drum as well.

With help from Bérard, Buser, and others, the team of Gordon, Webb, and Wolpert have streamlined their proof and found many other examples of sound-alike drums. Triangles turn out to work as well for building the drums as half-crosses (though seven seems to be a magic number). And not all the drums are as intricate as the first pair; some of them have as few as six sides.

The answer to Kac's question closes the book on one problem, but it raises new issues that should keep geometers busy for a while. For example, by showing that you can't hear every property of a drum, the discovery opens the question of just how many properties of a drum—besides its area and perimeter—really are "audible." Then there's the question whether more than two drums can produce the same set of sounds. That would be the ultimate nonconformist's nightmare: Say, everyone marches to the beat of a different drummer, but all the drums sound the same. ■ BARRY CIPRA

Controlling Chemical Reactions With Laser Light

Recent advances in laser technology are giving chemists the ability to enhance the breaking of specific chemical bonds

EVER SINCE THE DAYS OF THE ALCHEMISTS, synthetic chemists have had a straightforward goal: increase the yield of the desired product in a chemical reaction, while minimizing the formation of unwanted byproducts. But while this goal may sound straightforward, chemists have generally had to rely on relatively crude means to accomplish it. For example, they tinker with external variables, adjusting the temperature or pressure, or they change the composition of the solvent in which the reaction is run. But they've not been able to go right to the heart of a reaction to bend it to their will—until now.

Within the past few months, thanks to recent progress in laser technology, three independent research teams have shown that they can influence the course of chemical reactions by using laser light as a source of energy to facilitate the breaking of specific chemical bonds. The research so far has been done only with simple model systems, and the researchers are not yet willing to speculate about any eventual practical applications. But says theoretical chemist David Tannor of Notre Dame University: The work not only "illustrates a very good interplay between theory and experiment, but it opens the potential for vast amounts of control of chemical reactions. Advanced laser technology is allowing this discipline to take off."


ALBERT DADIAN

Tuning lasers to specific bonds. *Richard Zare.*

One of the advances Tannor is referring to is lasers, first developed about 5 years ago, that can produce extremely brief light pulses, lasting just femtoseconds—or millionths of a billionth of a second. In 1985, when he was a postdoc with Stuart Rice at the University of Chicago, Tannor had in fact proposed that if such ultrafast lasers became available they could be used to guide chemical reactivity. The reason: The duration of the laser pulses would match that of the key events determining chemical reactivity, such as the periodic stretching and relaxation of the bonds holding atoms together in molecules, which also take place in a matter of femtoseconds. The ability to time laser pulses to deliver energy to reacting molecules at just the right moment would mean that modern-day alchemists could either enhance or inhibit the breaking of particular bonds, thus enabling them to direct the course a reaction takes.

That Tannor was prescient is shown by an experiment, reported in the 2 January issue of *Nature*, in which Caltech chemist Ahmed Zewail and his colleagues used ultrafast laser pulses to control the reaction between molecular iodine (I_2) and xenon that produces xenon iodide. Zewail and his team began with the knowledge that the two iodine atoms in I_2 normally vibrate back and forth, periodically stretching the bond holding them together from 2.5 Å to 5 Å and then contracting it back again.

The idea was to exploit this motion to enhance the reaction between I_2 and xenon by first pumping one pulse of laser light into the reaction mixture to further excite the vibrational motion of the iodine atoms. This first pulse did not provide enough energy to break the bond between them and promote the formation of xenon iodide, however. That required the input of a second pulse of laser light, the timing of which is critical.

The researchers found that the xenon iodide yield was highest when the two pulses were delivered simultaneously; it dropped sharply when the second pulse was delivered 350 femtoseconds after the first; and went back up again when the interval between the two pulses was 700 femtoseconds. This shows that the best result comes when the follow-up pulse is timed to unload its energy when the bond between the two iodine atoms is stretched to the fullest.

In the Zewail team's work, the second pulse was used to enhance the effect of the first on the I_2 bond. But now another group, led by physical chemist Graham Fleming at the University of Chicago, has taken the method a step further, showing that it's also possible to use a second pulse to cancel the