Music of the Spheres

It took Wu-Yi Hsiang 100 pages to show how three-dimensional space can best be filled with spheres—thereby proving what Johannes Kepler didn't bother to

WHO SAYS TEACHING AND RESEARCH don't mix? Last spring Wu-Yi Hsiang agreed to develop a course in classical geometry at the University of California at Berkeley. The first thing Hsiang did was to pick the oldest, hardest, unsolved problem in the subject as a way of gearing up for the course. The second thing he did was solve it.

That's the upshot of a 100-plus-page manuscript that Hsiang is now refining before sending it out for the scrutiny of his fellow mathematicians. If his proof holds up under their gaze, it will spell the end of a nearly 400-year-old quest—and may set off a geometric goldrush.

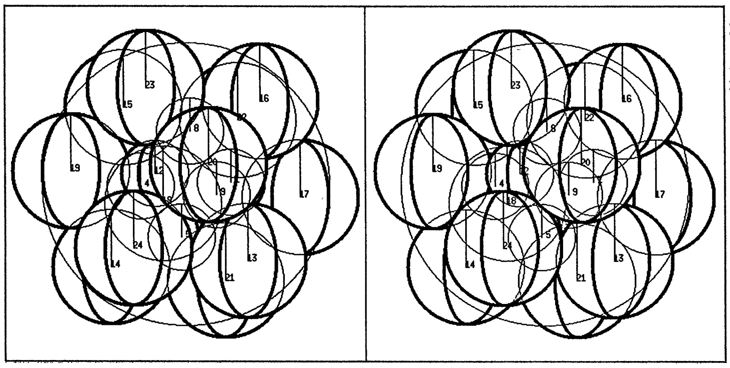
The puzzler Hsiang tackled is known as the sphere-packing problem. It addresses the age-old question: How many bowling balls can you fit in a broom closet? More mathematically, sphere-packing is concerned with finding a way to fit identical, nonoverlapping spheres together so as to fill the largest possible fraction of three-dimensional space—and then proving that you can't do better.

Says Hsiang: "I got hooked on it. The more I thought about this problem, the more beautiful it appeared."

While sphere-packing can be taken as a purely theoretical exercise (Hsiang's approach), it does have real world implications. Problems related to sphere-packing arise in areas such as radar antenna design and x-ray tomography, and higher dimension versions of the theory play a crucial role in the design of telecommunication systems.

But the real attraction to geometers is the raw challenge. "It's one of those problems that tells us that we're not as smart as we think we are," says Doug Muder, a mathematician at Mitre Corp. in Bedford, Massachusetts, who has also tried his hand at sphere-packing. "If we really understand three-dimensional space, and if we really understand spheres, then we should have no trouble solving a problem like this. But we do have trouble. Maybe after we see a solution, we'll start to understand something about space and spheres."

As often happens in mathematics, the answer to the question of how many spheres can be packed into a given space had been known—or at least suspected—for a long time. Indeed, Johannes Kepler figured it out in 1611 by pondering the spherical "loculi" of pomegranates: The densest packing occurs when spheres are crammed together in what's known as the face-centered cubic lattice arrangement. This is the arrangement that forms naturally when you build a pyramid out of pool balls, with the bottom layer held in the standard triangular rack. In this packing, the spheres fill approximately 74% of the space ($\Pi/\sqrt{18}$, to be precise).



Nelson Max

Kissing cousins. *Stereographic projections from four-dimensional space onto two dimensions of 24 spheres "kissing" a 25th. The projections were made by Nelson Max at Lawrence Livermore National Laboratory.*

The problem is that although Kepler had the answer, he didn't bother to prove it. And, until Hsiang started preparing for his geometry class, no one else had been able to. Surprisingly, Hsiang's proof uses none of the esoteric theories of modern mathematics. Instead, he employed venerable techniques of spherical geometry, vector algebra, and calculus to show that Kepler's proposed arrangement is correct. However, the proof is very complicated and extremely long. (One hundred-page papers aimed at a single theorem are rare in mathematics.) Which poses another problem: Who's going to take the time to check Hsiang's work?

The most natural candidates, says Muder, are mathematicians who have taken a crack at the same problem. And much of Hsiang's proof is likely to be familiar territory to them. "In a 100-page paper there may only be 10 to 20 pages that another expert will really need to go through line by line," Muder says. Thus how long the proof is, is less important than how it's put together.

"Length is not the issue," Muder emphasizes. "*Shogun* and *Finnegans Wake* are both long novels, but *Shogun* is a page-turner and *Finnegans Wake* is a semester-long project. The reason *Shogun* is so much easier is that on any given page it's easy to figure out what's going on and what it has to do with the overall plot. It's the same in a math paper. If Hsiang has written his paper so that the basic structure of the argument is well laid out and so that it's clear how each individual calculation fits into that structure, 100 pages should be no problem."

Others will likely be drawn in by a kind of goldrush effect: If Hsiang's proof has struck pay dirt, there are more mathematical nuggets waiting to be sieved out by the same techniques, and eager geometers will pore over Hsiang's proof like prospectors examining a geological report. And Hsiang himself has identified several "natural" problems in spherical geometry that may make up the lode that draws the prospectors.

One is sphere-packing in higher dimensional generalizations, where the problem remains unsolved. Another is the "kissing sphere" problem: How many identical nonoverlapping spheres can you fit around a single, central sphere? In three dimensions, the answer is 12, while in four dimensions, the answer could be 24 or 25 (the smart money is on 24). Even less is known about dimensions higher than 4—except for happy accidents in 8 and 24 dimensions, where the kissing number is known.

Work on these related problems is already under way. Nelson Max, a mathematician and computer graphics expert at Lawrence Livermore National Laboratory, recently showed that the single known example of 24 kissing spheres in four dimensions is "rigid," meaning that the spheres can only rotate as a single unit around the central sphere. Max has also written computer graphics programs for visualizing sphere-packing scenarios in four dimensions. Max's high-tech wizardry contrasts with Hsiang's methods, Max says: Hsiang works with a pocket calculator.

Hsiang has been lecturing on the proof and has sent out preliminary versions. He plans to have a final draft ready soon for his colleagues to look at. "Then it's their turn," he says. Unless someone quickly finds a fatal flaw in Hsiang's proof, it will probably take a year or more before the mathematical community is convinced one way or the other. In the meantime, Hsiang has a geometry class to teach. ■ BARRY CIPRA