Computer Search Solves an Old Math Problem

A math problem with roots that reach back 200 years has been solved by means of a massive computer search

A MATHEMATICAL PROBLEM with roots in the 18th century has been solved by a combination of intricate analysis and several thousand hours of computer time. A group of researchers in the computer science department at Concordia University in Montreal has proved that a certain type of "finite geometry" known as a projective plane of order 10 cannot exist. Their proof fills a gap that has been open for 40 years.

A finite geometry is a system of finitely many objects, called "points," with various subsets designated as "lines." A projective plane is a finite geometry obeying four rules. First, each pair of points belongs to exactly one line. Second, each pair of lines intersects at exactly one point. Third, there exist four points no three of which belong to the same line. And fourth, there exist four lines no three of which intersect at the same point.

The rules have an appealing symmetry: if the words "point" and "line" and the phrases "belong to" and "intersect at" are interchanged, there is no change in the rules. Consequently any theorem about points and lines has a "dual" theorem about lines and points.

The order of a projective plane tells how many points belong to each line (or, symmetrically, how many lines intersect at each point). It turns out that each line must contain the same number of points. Mathematicians then say that a projective plane is of "order n" if each line contains $n + 1$ points. (There is actually a sensible reason for this: deleting the points in any one line leaves an "affine" plane with n points per line. An affine plane is a geometry satisfying the "parallel postulate" that through any given point not on a given line there is exactly one line parallel to the given line.) A projective plane of order n has exactly $n^2 + n + 1$ points.

Which numbers n can occur as the order of a finite projective plane? There are relatively simple constructions of projective planes whose order n is a prime number or a power of a single prime, such as 8 or 81. A long-standing conjecture is that these are the only possibilities—that is, numbers that have more than one prime divisor, such as 6, 10, 12, 14, and so forth, cannot be the order of a projective plane.

In 1949, R. H. Bruck and H. J. Ryser proved a big chunk of this conjecture. They proved that if the remainder of n divided by 4 is 1 or 2 and if n cannot be written as the sum of two square integers, then n cannot be the order of a projective plane. This condition, strange as it sounds, takes care of infinitely many cases, including 6, 14, 21, and 22. However, it leaves open the status of numbers such as 10, 12, 15, 18, and 20.

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The Concordia group's new result confirms the conjecture for $n = 10$. Clement Lam, Larry Thiel, Stanley Swiercz, and John McKay carried out a case-by-case analysis of implications that the existence of an order 10 projective plane would have. Their analysis, based on work that was done in 1970 by F. J. MacWilliams and Neil Sloane of Bell Labs and John Thompson of Cambridge University, led to predictions that could be checked by computer in a reasonable amount of time. (In principle the result could be proved by simply considering all possible "geometries" consisting of 111 11-point lines in a set of 111 points and showing that none of them satisfies the rules of a projective plane, but such a "simple" proof would involve impossibly large amounts of computation.)

Even though it settles the conjecture for only one number, the computer proof is more difficult than Bruck and Ryser's. It required several thousand hours of computer time to search through more than a trillion separate cases. Lam and McKay began the first leg of the computation at Concordia in 1979. The final leg was finished on a Cray supercomputer at the Institute for Defense Analyses in Princeton, New Jersey. The Cray computation began in September 1986, running part time on a lowest priority basis, and was completed at the end of November 1988. Lam estimates that the proof involved "about two orders of magnitude more computing power" than Wolfgang Haken and Kenneth Appel's 1976 proof of the famous Four Color Theorem.

Although the proof depends on massive amounts of computation, it is far from a "brute force" approach, according to Lam. The Concordia group investigated different ways of attacking the problem, and made estimates on the amount of computation involved in each assault. "The final program we wrote is based on a lot of experimentation," Lam says. "If we had just written a program without doing the estimation, we'd probably have a program that was too long to be run anywhere."

The techniques developed in the course of proving the conjecture for $n = 10$ have proved helpful in settling a related problem of identifying the different projective planes of order 9 (there are four of them). Lam however doubts that they will be useful for proving the conjecture for $n = 12$ without substantially new ideas. If the techniques were applied as is, the computation for an $n = 12$ proof could exceed the $n = 10$ proof by as much as ten orders of magnitude.

Finite geometry is a relatively young subject, with much of its current motivation stemming from applications to combinatorial design and coding theory. However, some of the crucial questions can be traced back to over 200 years ago, when computational power was somewhat harder to come by. In around 1780, Leonhard Euler posed a simple-sounding problem: there are six regiments of six soldiers each, with six separate ranks within each regiment. Arrange the 36 soldiers in a six-by-six square so that each row and each column has a representative of each regiment and a representative of each rank. A four-by-four version of Euler's problem can be played with the aces and face cards from a deck of cards: arrange the cards in a four-by-four square so that each row and each column has a card of each suit and a card of each value. The main difference is that this version can be solved whereas Euler's problem cannot.

The general problem is formulated mathematically in terms of "orthogonal Latin squares." Given n distinct objects—such as letters of the alphabet or simply the numbers 1 through n—a Latin square of order n is an arrangement of the objects in each row of an n-by-n square such that each column also contains the complete set of objects. Two Latin squares are called orthogonal if every possible pair occurs when one square is laid on top of the other.

There is a close connection between finite projective planes and systems of mutually orthogonal Latin squares. (A set of Latin squares is "mutually orthogonal" if each pair of squares in the set is orthogonal.) For each projective plane of order n there is a system of $n - 1$ mutually orthogonal Latin squares of order n, and vice versa. The correspondence is elementary, though not completely obvious. Essentially, each Latin square corresponds to a set of n parallel lines in the affine geometry associated with the projective plane. It is fairly easy to prove that a system of mutually orthogonal Latin squares cannot have more than $n - 1$ members. This maximum size occurs only when n is the order of a projective plane.

Euler's problem amounted to asking for a pair of orthogonal Latin squares of order 6. It turns out to be relatively easy to prove that such pairs exist when the order of the squares is odd or evenly divisible by 4. Euler conjectured that no such pairs exist in the remaining case, when n is even but not divisible by 4.

The case $n = 2$ is trivial. Euler's conjecture for $n = 6$ was confirmed in 1900: there is no solution to the 36-soldier problem. However, the rest of the conjecture was demolished in 1959, when E. T. Parker, R. C. Bose, and S. S. Shrikhande showed that there is at least a pair of orthogonal Latin squares for each order other than 2 and 6.

Except for prime-power orders (corresponding to the known projective planes), it remains unknown how many mutually orthogonal Latin squares there can be of a given order. The Concordia group's result implies that there can be at most eight mutually orthogonal Latin squares of order 10, but so far no one has found even three. Lam thinks it is a hard problem. "I suspect that trying to find three Latin squares [of order 10] is even more work than what we have done," he says. ■ **Barry A. Cipra**

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Extinction Imminent for Native Plants

Some 253 native American plants are so imperiled that they are likely to become extinct within 5 years, and another 427 will probably vanish by the end of the century, according to a new study released last week. Nearly 8% of these 680 critically endangered plants are already thought to be extinct in the wild and are preserved only in arboreta and botanical gardens.

But compared with cute and cuddly endangered animals, plants have received relatively little attention and few conservation funds. This survey, conducted by the Center for Plant Conservation, a consortium of 19 botanical gardens and arboreta, was designed to help target conservation efforts to those species that are most in need.

Previous surveys have revealed that more than 10% of the nation's 25,000 species, subspecies, and varieties of native plants are at risk of extinction, mostly from habitat loss to agriculture or urbanization. What this survey did, for the first time, was to estimate the imminence of extinction—in other words, how much time is left for various species.

"We must know how fast the clock is ticking," said Donald Falk, executive director of the Center for Plant Conservation at a press conference at the Smithsonian Institution where the survey was released. "The survey tells us where we have to go first and how quickly we must work to save species before they become extinct."

Nearly three-fourths of the nation's critically endangered plants exist in just five states and territories: Hawaii, California, Texas, Florida, and Puerto Rico. As soon as the survey was completed, the center launched an emergency conservation program focusing on those five regions. Falk said a program to rescue all the plants at risk of extinction in this century would cost \$10 to \$15 million.

Center for Plant Conservation

Lesquerella pallida, *a bladderpod plant with delicate white flowers, had not been seen for 100 years until small populations were found at four sites in Texas.*

In conducting the survey, the center asked 89 botanists and horticulturists around the country to rank some 800 threatened and endangered plants according to whether they are likely to go extinct within 5 years, 10 years, or to survive beyond 10 years. The list was drawn from data from The Nature Conservancy and the U.S. Fish and Wildlife Service. The experts also added to the list any additional plants they considered at high risk.

Of the 680 plants likely to go extinct by the end of the century, only 91 are now maintained in cultivation or seed storage. Over the next 3 years the center plans to bring specimens of the other plants into the National Collection of Endangered Plants, a living collection of rare plants housed in 19 botanic gardens and arboreta around the country. The specimens can then be used for research and for propagation to enhance wild populations. For some plants, arboreta may be the only place they survive.

Some of the plants at highest risk include:

- A pretty Hawaiian shrub, *Hedyotis parvula*, which has shiny leaves and clusters of waxy white and pink flowers. It is known in the wild from just one specimen that has been found growing at the base of a cliff on the island of Oahu.
- The large-flowered amsinckia, or *Amsinckia grandiflora*, is known from just one population growing on Army property in California. It has brilliant yellow-orange flowers.
- A bizarre prickly pear cactus with bright red flowers, *Opuntia spinosissima*, survives on private land on one small island in the Florida Keys. Only six plants remain.
- Peter's Mountain mallow, or *Iliamna corei*, a shrub with stunning pink flowers, is found at only four sites in Virginia.

■ **Leslie Roberts**