Number Theory Connections

An unusual meeting honoring mathematician John Tate had as its theme connections between number theory and diverse areas of mathematics

When Harvard mathematician John Tate turned 60 recently, some of his students and collaborators planned a most unusual meeting to commemorate the occasion. The conference organizers, Barry Mazur of Harvard, Jean-Pierre Serre of the College de France, and Stephen Schatz of the University of Pennsylvania, invited Tate's former students, his collaborators, and mathematicians with interests similar to Tate's to Harvard from 29 April to 3 May. But only one formal talk, by John Coates of the University of Paris at Orsay, was prearranged. The program for the rest of the meeting was decided in two plenary sessions in which the participants decided whom they wanted to hear speak, what they wanted to hear them speak about, and in what order.

The speakers were not necessarily asked to discuss their own work—the only criterion was that they speak of as many new and important developments as possible. Only three talks a day were scheduled, leaving lots of time for informal discussions. Once the participants decided on the first 2 days' speakers and their topics, the newly scheduled speakers dashed off to their hotel rooms to prepare their lectures. The second planning session was held midway through the meeting to determine the schedule for the last 2 days.

The conference organizers say that this style of conference is very rare in the United States but that it is similar to meetings organized by F. Hirzebruch of the University of Bonn and known as the Mathematische Arbeitstagung. These meetings have been held almost every year for the past quarter century in Bonn.

In the opening planning session for the Bonn conference, any subject in mathematics is fair game, and although this was also the case at the conference at Harvard, the talks tended to be on number theory and its connections to other areas of mathematics, which is Tate's special interest.

Two talks at the meeting were a forgone conclusion. Very recently, two of the meeting participants independently made progress on a long-standing conjecture of Tate's. Gerd Faltings of Princeton solved the problem while Jean-Marc Fontaine of Grenoble, working with William Messing of the University of Minnesota, proved a refinement of Tate's conjecture using another method. Faltings and Fontaine were at the meeting and, naturally, they were asked to speak about their work.

The newly resolved conjecture concerns a theory based on number systems called p-adic fields. These fields sound strange but, Tate remarks, they are, "actually no more abstract than the real numbers [the ordinary number line]. It is just that the real numbers are used to measure the variation of physical quantities and so everyone can visualize the real number line. Logically, the p-adics are on exactly the same footing as the real numbers." Yet the p-adics are not of interest merely because they are an odd analog of the ordinary numbers. It turns out that they frequently are just what is needed to analyze certain solutions of equations.

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To build the p-adics, mathematicians start with ordinary integers, which are whole numbers such as 1, 2, and 3. Then they choose a prime number, p. They say that two numbers are p-adically "close" if their difference is divisible by a high power of p. So if p is 5, two numbers are 5-adically close if their difference is divisible by a high power of 5. That means that 6 is close to 1 because 6–1 is 5, but that 26 is even closer to 1 because 26–1 is 25, or 5^2—an even higher power of 5. Still closer to 1 is 126 and closer still is 626.

P-adic numbers, says Mazur, "are limits of infinite sequences of ordinary integers whose terms get p-adically closer and closer. At first glance this seems very bizarre, but the usefulness of p-adic numbers comes in when we try to solve certain equations." For example, the equation might be $y^2 + y = x^3 - x$. "If you're looking for ordinary integers that are solutions to such equations, it can be quite hard," Mazur remarks. "But suppose you ease up a bit. Suppose you choose a power of a prime number p, say p^n, and look instead for x and y such that the two sides of the equation differ by a multiple of p." This turns out to be much easier than the original question. Any pair of integers x and y that is a solution to the original equation is a solution to this easier problem as well. But for certain important classes of equations, called indefinite quadratic forms, mathematicians have shown that if they can find solutions of these easier problems for all powers of prime numbers, then the original equation has integer solutions. So, says Mazur, looking at these easier problems, "is not a completely ridiculous thing to do."

The p-adics come in because solving the easier form of these equations for all primes is the same as finding a solution in the p-adics. For this reason, Tate notes, mathematicians find it convenient to work in the p-adics. And, adds Mazur, "there are certain geometrical theories involving the real numbers that have close analogies in the p-adics where the geometry is less evident." One of these is Hodge theory, which seems crucially built on real numbers. But Tate formulated a conjecture saying that an analogous theory holds in the p-adics. This is the conjecture that has now been resolved.

Other speakers talked on recent results in factoring (*Science*, 19 April, p. 310), on the solution to the class number problem (*Science*, 7 October 1983, p. 40), and on a number of as yet unsettled conjectures. For example, Theodore Chinburg of the University of Pennsylvania gave a talk on "Galois structure and Stark's conjecture."

Stark's conjecture connects number theory to analysis and involves an abstraction of the concept of a unit integer. In the ordinary integers, the only units are 1 and -1. These are the whole numbers whose reciprocals are also whole numbers. But, says Mazur, "the more general definition of a unit comes into play when mathematicians look at what they call algebraic integers. These are expressions that act like integers but are a bit more complicated."

An example of a system of algebraic integers is the collection of all numbers of the form $A+B\sqrt{2}$, where A and B are ordinary integers. You can add and multiply numbers of this form and the answer will always be a number of the same form. The units of a system of algebraic integers are those algebraic integers whose reciprocals are also algebraic integers. So, for example, in the system of algebraic integers of the form $A+B\sqrt{2}$, $1 + \sqrt{2}$ is a unit since its reciprocal is $-1 + \sqrt{2}$. In fact, Mazur notes, any unit in this system is either a power of $1 + \sqrt{2}$ or a power of its reciprocal or the negative of any of these.

Other systems of algebraic integers can be more complicated and their units can be difficult to determine. Harold Stark of the Massachusetts Institute of Technology and the University of California in San Diego conjectured that there is an amazing relationship between certain expressions involving the logarithm of these units of algebraic integers and the behavior of particular functions, called non-Abelian L functions, L(s), at the point where s is 0. He proposed that you can start with these functions from analysis, L(s), and get expressions involving the logarithms of units of algebraic integers and, in special cases, get the logarithms of the units themselves. What is surprising is that Stark thereby connects analysis and number theory.

Stark showed that his conjecture is true in some cases. And although no one yet has an inkling of how to prove the conjecture in general, work on it has suggested several new mathematical ideas.

Tate recently wrote a book on these developments and proved the conjecture in another special case. Tate's proof then led to new questions about the units of algebraic integers and Chinburg reported on recent research on these questions.

Other talks at the meeting were on problems relating number theory to algebra, algebraic geometry, and analysis. The theme, says Mazur, "is connecting what once seemed to be the unconnectable"—a possiblity that is bound to be exciting.—**GINA KOLATA**

When Are Viscous Fingers Stable?

Recent research concludes that a single, stable finger can form when a lower viscosity fluid pushes against one of higher viscosity

Interest in viscous fingering, a decades-old problem in the fluid dynamics literature, has taken on new life in the last few months. Activity is on two fronts, which already look as though they are quite closely related.

In the first area, theorists find they can now explain the persistence over long times of the distinctive finger patterns, although a complete quantitative description of the shapes is not yet in hand. At the same time, experimentalists and theorists have been jointly exploring the limits of the fingers' stability and find it to be not unlimited. A low surface tension at the interface between the two fluids, a large interface velocity, and fluctuations or noise at the interface all degrade the stability and give rise to distortions in the fingers in both numerical simulations and experiments.

The second area concerns the recently fashionable topic of fractal behavior in physical systems. Fractal objects have a property called self-similarity—that is, they have similar features at all length scales and therefore look the same at all magnifications—and are characterized by an effective fractional dimension, rather than the integer 1, 2, and 3 of curves, surfaces, and volumes.

Self-similarity is a kind of symmetry, in this case invariance under a change in length scale. This sort of symmetry was a crucial ingredient in the development of the theory of phase transitions (critical phenomena) over a decade ago, and more recently it has figured in the behavior known as chaos in nonlinear dynamical systems (*Science*, 5 November 1982, p. 554). Now scientists hope that it will play a similarly powerful role in understanding other physical phenomena. The recent observation of fractal viscous fingers, to be discussed in a second article, is therefore causing much excitement.

The question of the stability and shape of viscous fingers is part of a more general domain of inquiry called pattern formation. Physical systems that evolve under conditions far from equilibrium often take on characteristic shapes or patterns that are governed by a balance between competing forces acting on the system during its growth. Fluids are an especially fertile ground for such processes, as exemplified by the wonderful patterns of whorls and swirls that occur at flow rates between the laminar and completely turbulent regimes (*Science*, 8 July 1983, p. 140).

The mathematically simplest example and hence the prototype whose understanding should help with the solution of more complex problems in pattern formation is the single finger that can occur when a lower viscosity fluid pushes against one of higher viscosity.

A key event in the modern history of viscous fingering was a 1958 publication by Philip Saffman (now at the California Institute of Technology) and the late Sir Geoffrey Taylor of the University of Cambridge describing the displacement of a viscous fluid, such as oil, by a less viscous fluid, such as water, in a cell comprising two closely spaced flat plates. This flat-plate configuration is called a Hele-Shaw cell, after the British engineer J. H. S. Hele-Shaw, who invented it in 1898. As a kind of two-dimensional wind tunnel for liquids, it was useful for studying fluid flow past obstructions, such as ship hulls.

Saffman and Taylor were more intrigued, however, that the mathematics of their two-fluid experiment was also the same as that for flow in porous media, a problem of great interest to petroleum engineers and civil engineers. For example, one of the methods of enhancing the productivity of an oil field is by pumping water or carbon dioxide gas into the ground through one well in order to force more oil to flow to neighboring wells. The formation of fingers of water or gas in the oil, as was observed to occur in laboratory models, plainly affects the efficiency of the recovery, although there are probably other equally important factors in real oil fields, such as faults in the rock.

Whether or not the Hele-Shaw cell really does provide a model system for fluid flow in porous media, the distinc-