Does Gödel's Theorem Matter to Mathematics?

The recent discovery of two natural but undecidable statements indicates that Gödel's theorem is more than just a logician's trick

In the 1930's, Kurt Gödel shook the world of mathematics by showing that there are statements in every logical system whose truth or falsehood simply cannot be determined by staying within the system. And if you try to fix up a logical system by calling the undecidable statements axioms and thereby declaring them to be true, new undecidable statements will crop up.

This result made mathematicians wonder about many of the famous unsolved problems that plague them. Could it be that some of these problems are undecidable? "What people would really like is to take a big unsolved problem like Fermat's Last Theorem and show it is undecidable. That would be spectacular," says Joel Spencer of the State University of New York in Stony Brook.

But, so far, that has not happened and mathematicians have engaged in philosophical debates over whether it ever will, whether Gödel's theorem applies to statements that matter. Many think it does not. Craig Smoryński, a logician at Ohio State University, remarks, "It is fashionable to deride Gödel's theorem as artificial, as dependent on a linguistic trick." Logician Robert Solovay of the University of California at Berkeley adds, "The feeling was that Gödel's theorem was of interest only to logicians."

A few years ago, however, two logicians found an example of a "natural" statement, involving only finite quantities, that cannot be proved true within the normal axiomatic structure of finite mathematics. In a sense, this statement just missed being provable. Now, another logician has found an even more "natural" statement that cannot be proved true in an even stronger system of axioms. The proof of this second statement requires structure far beyond the mathematical system used for finite quantities, raising it to a much higher level of undecidability. These two results are leading a number of mathematicians to believe that Gödel's theorem does in fact apply to problems that matter.

The first of these undecidable statements was discovered by Jeff Paris of Manchester University and Leo Harrington of the University of California at Berkeley. The statement involves combinatorics and is "natural" because it is not the sort of concocted statement that only logicians would devise. "The Paris-Harrington theorem looks like a natural mathematical question with no trace of logic about it. That's what's spectacular—it is natural and combinatorial in character," says Solovay.

The theorem is a statement in Ramsey theory, which is a branch of mathematics dedicated to the proposition that "complete disorder is impossible," according to Ronald Graham of Bell Laboratories. If you choose a big enough set, you are bound to find structure in it. The question is, however, how big must the set be? In the case of the Paris-Harrington theorem, the size of the set grows so large so quickly that the function describing its growth simply cannot be shown to be well defined in Peano arithmetic, which is the ordinary axiomatic system used in mathematics to talk of finite things. Peano arithmetic, says Spencer, is "the accepted bedrock of mathematics."

A special case of what is known as Ramsey's theorem is the party problem: How many guests must you have at your party to be assured that a certain number of them either all know each other or all are strangers to each other? If you want to be sure that at least three guests are mutual acquaintances or mutual strangers, you must have at least six people at your party. If you want four guests all to know each other or all to be strangers you need at least 18 people at the party. But no one knows the minimum number of guests you need at the party to guarantee a similar group of five. The number is somewhere between 42 and 55. Says Graham, "It is hopeless to try and compute the exact number. It is way beyond our present computing power."

The Paris-Harrington theorem is a slight variation of Ramsey's theorem. According to Ramsey's theorem, if you have an infinite set and you assign a color, say red or blue, arbitrarily to each pair of members of the set, then you can find an infinite subset, all of whose pairs are red or all of whose pairs are blue. More generally, if you pick numbers r and k and if you have an infinite set and you assign one of r colors arbitrarily to each k-element subset of the set, then there is an infinite subset, all of whose k-tuples have the same color.

Paris and Harrington devised a finite version of Ramsey's theorem. They started out by defining a "large" set of integers to be one that has at least as many elements as its smallest integer. For example, the set 3, 15, 25, 26, is "large" because it has at least three elements. The set 100, 102, 104, 106, 108 is not "large" because it has fewer than 100 elements. Then Paris and Harrington showed if you take a big enough set of integers and assign colors, such as red or blue, to each pair of integers you can find a "large" set, all of whose pairs are red or all of whose pairs are blue. Or, more generally, if you choose r and k and assign r colors to the k-element subsets of your initial set, you can always find a "large" subset, all of whose k-tuples are the same color.

How big must your original set be? It depends on how many colors and how you partition the subsets, but Solovay found that the lower bound on the size of the set grows so fast that it isn't even well defined in Peano arithmetic. Says Graham, "The lower bound for how large the initial set must be grows fast. It is hard to grasp how fast it grows. It grows so quickly that the numbers somehow begin to lose all meaning."

The way the lower bound grows is analogous to the way a function, called the Ackermann function, grows. This is a function of two variables that is recursively defined: $f(a, b) = f[(a - 1), f(a, b - 1)]$ where $f(1, b) = 2b$ and $f(a, 1) = a$ for a greater than 1. With this function,

$$f(3, 2) = 2^{2^2} = 16,$$
$$f(3, 4) = 2^{2^{2^{2^2}}} = 2^{65536},$$

a number with more than 19,000 digits. (When evaluating towers of exponents, mathematicians work from the top of the tower down.) The term $f(6, 6)$ is so large that if you wanted to evaluate it you couldn't write it on a piece of paper. And these are just the initial values of the function—the values that are very close to the origin.

Paris and Harrington used model theory, a standard method of mathematical logic, to show that their theorem is undecidable in Peano arithmetic. Harrington explains that they produced two models for Peano arithmetic—two equivalent sets of axioms. In one of these models, the theorem was true and in the other it was not true, indicating that the theorem is undecidable. The analogy is with the

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axioms for geometry. In Euclidean geometry, one model, parallel lines never meet. In non-Euclidean geometry, a different model, they can meet.

After proving that the Paris-Harrington theorem is undecidable in Peano arithmetic, Harrington wrote a letter to Solovay noting that they had obtained this result but not saying how they got it. Solovay and his colleague Jussi Ketonen then devised their own proof that the theorem is undecidable, a proof that is more combinatorial in nature.

Solovay and Ketonen showed that because the lower bound on the size of the initial set grows so fast, you need a structure just beyond Peano arithmetic to prove it is well defined and thus to prove the Paris-Harrington theorem.

Peano arithmetic contains all the integers up to "infinity," which is denoted ω. Then, after the first copy of the integers, it continues with the terms $\omega + 1$, $\omega + 2$, and so on up to 2ω. (Spencer likens the system to the children's book *On Beyond Zebra* which goes on after the alphabet ends at *z* for zebra.) But Peano arithmetic does not end at 2ω. It continues to

$$\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$$

Yet even the exponential tower of ω's is not large enough to deal with the Paris-Harrington theorem. What is needed is ϵ_0, defined as the limit to which $\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$ converges. Asked how you know that the tower of ω's converges, Spencer replies, "It takes a leap of faith."

Very recently, Harvey Friedman of Ohio State University found a second undecidable theorem, but his theorem involves a function that grows so fast that it dwarfs the function of the Paris-Harrington theorem. Even ϵ_0 is not enough to prove Friedman's theorem. "The Paris-Harrington theorem lies just barely beyond Peano arithmetic," says Spencer. "Friedman's theorem is much farther out."

Friedman's theorem is a finite version of a well-known result discovered by Joseph B. Kruskal of Bell Laboratories. Kruskal's theorem involves "trees," which are sets of points connected by lines and containing no cycles. Evolutionary biologists draw trees when they describe the ancestors of species and geneologists draw family trees.

Collections of trees can be infinite as well as finite and, if they are infinite, complete disorder is impossible, according to Kruskal's theorem. The theorem states that if you have an infinite collection of finite trees ordered in any arbitrary way, then at least one of those trees must fit into a later one so that the branches of the first fit inside those of the second. Kruskal's theorem forms the basis for a branch of combinatorics called "well-quasi-ordering."

To make a finite version of Kruskal's theorem, Friedman said that you don't need an infinite collection of trees. All you need is a sufficiently large finite collection of trees. How large is large? That is where the enormous function comes in. "It is *gigantic*. I mean it's *really* gigantic," says Graham. According to Smoryński, it is "the most rapidly growing computable function that has ever been described."

The observation that Friedman's theorem is far beyond the reach of Peano arithmetic demonstrates, to Spencer at least, that Kruskal's theorem is indeed a deep one. "Friedman's result bears this out since he shows that if you turn Kruskal's theorem into a finite theorem, the proof is beyond the normal methods of finite mathematics." Harrington is impressed by the extreme naturalness of Friedman's theorem. It is the sort of theorem, he says, that could have arisen in combinatorics with no reference to mathematical logic and undecidability. "I found it easy to convince myself that combinatorialists could have thought of this," Harrington says.

Friedman also demonstrated that if you take a large enough collection of finite trees and ask whether a finite tree of a particular size (as opposed to any arbitrarily chosen size) must fit into another tree, you can use Peano arithmetic to show that it must. But the proof requires an enormous number of steps. Friedman showed that if, for example, you want to prove that a tree containing ten nodes must fit into another tree, the proof would require more than

$$2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}\Big\}\text{ 1000 times} \quad \text{steps.}$$

Smoryński predicts that the enormous function Friedman has described is just the beginning. Friedman has only dealt with a weak form of Kruskal's theorem. He is now working on a finite version of the full form of Kruskal's theorem and, according to Smoryński, "when these results are finitized, they will yield functions that dwarf F [the function Friedman has so far described]."

The more natural but undecidable theorems that are found, of course, the more willing mathematicians are to believe that Gödel's theorem might apply to important results. The recent discoveries of the Paris-Harrington and Friedman theorems might also lead mathematicians to a greater appreciation of mathematical logic, including infinite objects such as ω^{ω} and ϵ_0. "It shows the mathematical reasonableness of these weird objects," says Harrington.

—**Gina Kolata**

Famous Large Numbers

Large numbers have an inherent fascination for mathematicians who sometimes compete among themselves to see who can write the largest number on a 3 by 5 card. And when enormous numbers come up naturally in proofs, they achieve a sort of notoriety.

Early in this century, the champion large number was the Skewes number, discovered by S. Skewes in his attempt to determine when the values of a function in number theory change from negative to positive. In 1933, he proposed his number as an upper bound on the solution. The number is

$$10^{10^{10^{34}}}$$

The Skewes number has been greatly superseded in recent years, and the current world's champion large number used in a serious mathematical proof is a number derived by Ronald Graham of Bell Laboratories. (Graham's number is in the *Guiness Book of World Records*.) The number is an upper bound on a combinatorial problem and even to write the number takes a special "arrow" notation. The notation $3 \uparrow 3$ means (3) (3) (3). The notation $3 \uparrow \uparrow 3$ means $3 \uparrow (3 \uparrow (3 \uparrow 3))$. The notation $3 \uparrow \uparrow \uparrow 3$ means $3 \uparrow \uparrow (3 \uparrow \uparrow (3 \uparrow \uparrow (3 \uparrow \uparrow 3))$.

Graham's number starts with $3 \uparrow \uparrow \uparrow \uparrow 3$—but that is just the top number in a tower of exponents. There are 64 layers of exponents in the tower, each of which is the number $3 \uparrow \uparrow \uparrow \uparrow 3$. Says Craig Smoryński of Ohio State University, "Now *this* is something that the mathematician of today regards as large."—**G.K.**