Topologists Startled by New Results

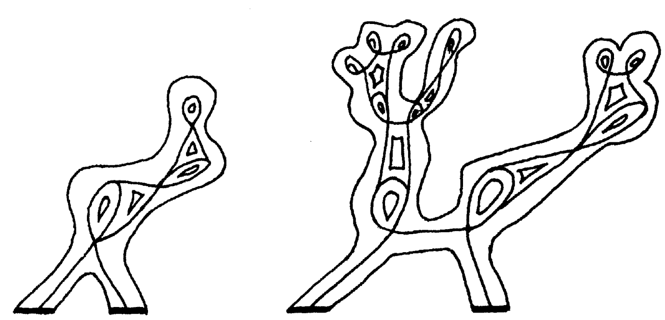
First a famous problem was solved, and now there is proof that there is more than one four-dimensional space-time

During the past year, the normally quiet field of topology has been shaken to its roots by a series of discoveries, starting with the solution to a long-standing problem and culminating in a finding so astonishing that mathematicians have difficulty believing it.

The discoveries were made by Michael Freedman of the University of California at San Diego and by Simon Donaldson, a second-year, 24-year-old graduate student at Oxford University who did the work as his Ph.D. dissertation under the direction of Michael Atiyah. The startling consequence of their work is an existence proof that there is more than one structure for ordinary four-dimensional space-time.

"I've known of this possibility for 3 months, and I still find it hard to believe," says topologist Robion Kirby of the University of California at Berkeley. Asked about the implications of the result for physicists, Kirby remarks, "I could daydream that the universe we live in is one of those peculiar structures rather than the one we always thought it was." Clifford Taubes, a mathematical physicist and junior fellow at Harvard, speculates, "It could have no physical meaning or it could mean that there are universes we can't talk to."

A three-stage tower (left) and its subunit (right). The three-stage tower is a basic construct in Freedman's proof of the four-dimensional Poincaré conjecture.



A number of researchers, while excited by the result, are advising caution in discussing it. Isadore Singer of the University of California at Berkeley, for example, says, "It seems that the result is correct, but it is hard for me to imagine. It goes contrary to the intuition and the experience of topologists, so one has to be a little bit cautious." Two meetings were held recently to discuss the result. Kirby, who just returned from one of the meetings, remarks that Donaldson's work, "doesn't look as hard as it once did. If there's a mistake it's a subtle one."

Last August, Freedman announced he had solved an old and famous problem—the four-dimensional Poincaré conjecture. Poincaré, a French mathematician who lived nearly 100 years ago, made his conjecture about the structures of three-dimensional objects. "The general problem in topology is how to find rules or procedures for recognizing spaces," says Freedman. "Topologists are not interested in the exact shape of a space—they are interested in the general form. One space is the same as another if all the points in one can be corresponded to points in the other without tearing." A circle and a line, for example, are not topologically equivalent because you would have to tear a circle to map it onto a line. However, a doughnut and a coffee cup—topologists' famous example—are equivalent. Imagine the handle of the coffee cup expanding and the cup shrinking until the cup becomes a doughnut. The point is that, to a topologist, it is the holes that are important and not the shape of the object. An old joke among mathematicians is that a topologist is someone who cannot tell a doughnut from a coffee cup.

What Poincaré conjectured is that, whenever any circle on a three-dimensional surface can be shrunk to a point, the surface is topologically equivalent to a sphere.

"Imagine a piece of string tied in a loop," Kirby says, "Put the loop on the surface and shrink it as much as possible." On a sphere, no matter where you put the loop, you can shrink it and it will not catch on anything. But that effect does not occur on structures that are not equivalent to spheres—such as doughnuts. The loop could get caught around the doughnut's hole as you try to shrink it to a point.

No one has ever resolved the three-dimensional Poincaré conjecture, and it remains one of the most famous open problems in mathematics. The last time any progress had been made on this problem was in 1959, when Stephen Smale of the University of California at Berkeley proved the conjecture for five dimensions and up. For this work, Smale won a Field's medal—mathematics' highest award. Freedman's solution to the four-dimensional version of the conjecture used all of Smale's ideas and more and had its origins in work, done a decade ago, by Andrew Casson, who is now at the University of Texas in Austin.

"Freedman took Casson's ideas and started to develop them," says Kirby. "He got a nice partial solution in 1978, but he continued to push the ideas." It took Freedman 8 years of concentrated work to get his result. Kirby remarks, "Freedman doesn't know when to quit. He would tell me and some others about his work. We would agree it was interesting, but we didn't think he could push it any farther; but he did."

Kirby and others in the field of topology were quite surprised when Freedman announced that he had proved the four-dimensional Poincaré conjecture. "I think it is one of the loveliest pieces of math I've seen. It has an element of originality. If Freedman hadn't done it, I don't think anyone would have done it for a long time," Kirby says.

But, in proving the conjecture, Freedman actually got a far broader result. He took a large step toward classifying all four-dimensional topological manifolds, which are spaces that look like ordinary

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Euclidean space at a point. Topologists divide manifolds into two types, called topological and differentiable. Topological manifolds can change abruptly as you move about the space; differentiable manifolds change smoothly.

Kirby explains, "If I stood at one point in a differentiable manifold and you stood at another point, and we each drew a set of coordinate lines about our points, my coordinates would look curved to you as though they had tangents, just as, in calculus, a graph of a differentiable function has tangent lines at each point. If we did the same thing in a topological manifold, it's hard to imagine how wiggly my lines would look to you. And yours would look very bizarre to me."

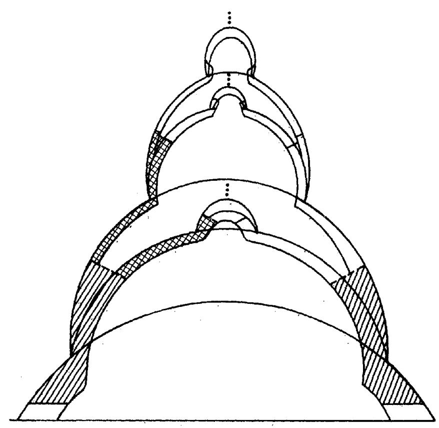
What Freedman proved, as a result of his work on the Poincaré conjecture, is that, although every differentiable manifold is also a topological manifold, not all topological manifolds are differentiable. This is a phenomenon that first occurs in four dimensions.

Freedman also constructed a particular topological four-dimensional manifold, called E8, that has eight two-dimensional holes. The Russian mathematician V. A. Rokhlin proved more than 30 years ago that no differentiable E8 exists, but topologists had wondered ever since whether a topological E8 exists. They knew there was no reason why it could not exist, but no one knew how to prove it was there. It is of special interest to topologists because it is a basic building block of higher dimensional manifolds. And the existence of a topological E8 is crucial to Donaldson's work.

While Freedman was working on manifolds and the Poincaré conjecture, mathematicians and physicists were using the vastly different techniques of differential geometry to study the existence of solutions to specific partial differential equations.

Differential geometry and topology are so unalike that specialists in the two fields use completely different language—and find it difficult to talk to each other. "They [topologists] think of pushing around balls of clay. They are not interested in local information, but they want to know how they can push things around and deform one space into another," says Taubes, whose work is in differential geometry. "I demand extra structure before I even sit down to work. I need a distance measure, for example. I think more in terms of an engineering drawing," Taubes remarks.

Taubes, as a theoretical physicist, drew on work by Singer, Atiyah, and Nigel Hitchin of Oxford. Mathematicians had defined what they call "connections," which, in a general sense, establish which direction in abstract spaces is horizontal. To physicists, these connections are the forces between subatomic particles. What mathematicians call "special connections" are the configurations of force fields that have the lowest potential energies. Since energy is conserved, these fields with lowest energies are the ones physicists expect to occur most often.



A diagram that Freedman calls "the design"

It is made up of three-stage towers and is the beginning of an infinite construction. The design, says Kirby, "is the most remarkable step in Freedman's proof of the four-dimensional Poincaré conjecture".

In the course of his work on partial differential equations, Taubes proved that, if two of the manifolds E8 were stuck together and could be smoothed, then the resulting space would have to have special connections. But it remained an outstanding open problem to determine whether what mathematicians call E8 + E8 could ever be smoothed out.

It was known that E8, itself, must be so convoluted that it could never be smoothed. Also, it was known from Freedman's work that, if there were only one possible Euclidean four-dimensional space—only one way to construct what physicists conceive as ordinary four-dimensional space-time—then E8 + E8 could be smoothed.

To everyone's surprise, Donaldson saw that Taubes' result could be used to study the problem of smoothing E8 + E8. But before he could do this he needed one further result derived by differential geometer Karen Uhlenbeck of the University of Illinois at Chicago Circle. Uhlenbeck showed that once it is known that a four-dimensional space, such as E8 + E8, has special connections, the space of special connections has a natural boundary which is the original four-dimensional space. Uhlenbeck says she "hadn't really thought of that as important, but the people at Oxford picked it up in a very innovative and startling way."

Donaldson says he first got the idea for his result last November. "It came out of other work I was doing in related problems." In the past few months, he put his proof together, convinced that although the result was surprising it was nonetheless true. "I believed it myself for quite some time because I couldn't see any way it couldn't be true," he says.

The proof that there must be at least two Euclidean four-dimensional spaces proceeds by contradiction. Although the mathematics is difficult and innovative, the bare bones of Donaldson's argument are as follows: Assume that E8 + E8 could be smoothed. Then, from Taube's theorem it follows that E8 + E8 has special connections. But if E8 + E8 has special connections, it follows from an extension of Uhlenbeck's work that E8 + E8 must have no holes in it. However, topologists know that E8 + E8 has 16 holes in it. Therefore E8 + E8 cannot be smoothed. If E8 + E8 cannot be smoothed, there must be more than one manifestation of ordinary four-dimensional space.

Taubes says it is too soon to say what Donaldson's result, if true, will mean to physicists, but it will certainly be of great interest to them. Uhlenbeck says the result "has to be related to work in partial differential equations," but, once again, it is premature to speculate on how the result will affect her field.

As for topology, Kirby speculates that Donaldson's result means, "Differentiable four-manifolds are much more complicated than we had any right to think. It makes my field more interesting and richer in its complexity." Atiyah, a geometer, says of his student's result, "I think it probably means that, instead of four-dimensional geometry getting to a stage of being completely understood, really it's just the other way around. We're just beginning to understand it. Four-dimensional geometry is much more interesting and complicated than we had any right to think."

What is particularly intriguing to Freedman and Kirby is that Donaldson's surprising result is about four-dimensional space—a space that is clearly of physical importance and a space that is not particularly abstract. "In four dimensions you can almost see things," Kirby says.—Gina Kolata