Aztec Arithmetic: Positional Notation and Area Calculation

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Summary. Texcocan-Aztec peoples in the Valley of Mexico used both picture symbols and lines and dots for numerical notation. Decipherment and analysis of mid-16th-century native pictorial land documents from the Texcocan region indicate that the line-and-dot system incorporated a symbol for zero and used position to ascribe values. Positional line-and-dot notation was used to record areas of agricultural fields, and analysis of the documentary data suggests that areas were calculated arithmetically. These findings demonstrate that neither positional notation nor the zero were unique to the Maya area, and they imply an equally sophisticated mathematical development among the Aztecs.

Systematic destruction of native written records by the Spaniards after the conquest of Mexico deprived modern scholars of much primary data important for determining the literary and scientific achievements of Mesoamerican civilization. A chance explanatory notation on an 18th-century native document enabled the German scholar, Ernst Förstemann, to begin in 1880 to decipher the arithmetical-calendrical portion of the Maya hieroglyphic system (*1*). As a consequence of Förstemann's work and that of his successors, basic aspects of the Maya arithmetical system are now understood. Their early use of a symbol for zero, for example, is one of the noteworthy achievements in Maya mathematical development. Maya scholars especially have long felt that other ethnic groups (Aztecs, Mixtecs, Zapotecs) lagged far behind the Maya in arithmetical capabilities (*2, 3*). However, it is well known that the Aztecs possessed a complex system of counting, which was expressed in hieroglyphic writing by numerous symbols. Furthermore, our recent decipherment of two early post-conquest census-cadastral documents from the Valley of Mexico demonstrates that Texcocan-Aztecs used a positional notation system. This has heretofore been attributed only to the Maya and consequently suggests the potential for equally sophisticated arithmetic among the Aztecs.

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Aztec Counting

The Aztecs had a great propensity for measuring and counting. Observing the market in their capital of Tenochtitlan, Hernán Cortés noted that "Everything is sold by count and measure . . ." (*4*). The Nahua language, spoken by the Aztecs, reflects the cultural importance of counts and measures. For example, the Nahua system of numerals makes use of classifiers that indicate the category of object counted. One classifier is used to denote round objects, such as eggs and fruit; another is used to count objects in which length is the primary feature, such as rope, walls, and irrigation ditches; still another denotes items that can be stacked, such as sandals and plates, or objects which are diverse or disparate from each other. Finally, some objects, such as cotton mantles, sheets of paper, or tortillas, are counted in groups of 20 (*5*). Proper counting in Nahua therefore requires knowledge of the class or category to which the object is semantically assigned.

As elsewhere in Mesoamerica, the Aztec arithmetical system was base 20. The term for 20 is literally translated "one count" (*cem-poalli*). The next place in the system is 20^2 or 400 (*cen-tzontli*), literally "one hairs." The largest place denoted by a special term is 20^3 or 8000 (*ce-xiquipilli*), literally "one bag." The integers between places are quantities to be added or multiplied. For example, 49 is expressed as *ompoalli onchiconahui* or "two counts and 9," and 500 as *centzontli ipan macuilpoalli* or "one hairs and 5 counts."

Symbolic systems for numerical notation varied in the Valley of Mexico at the time of the conquest. The one most widely associated with the Aztecs employed a set of picture symbols: a small circle for one, a flag for 20, a feather or tied bundle for 400, and a bag for 8000. Each denoted a cardinal place in the numeral system. Such a notational system would be cumbersome for use in arithmetical computation because of the complexity of drawing such figures as the bag. But, it has high visual impact in denoting quantities of commodities received in tribute, and much of the extant pictorial record of the Aztecs is concerned with tribute collection.

In addition to picture symbols, Texcocan-Aztecs used a positional line-and-dot system that had the advantage of writing efficiency, since it employed only four symbols that were easy to draw—a vertical line, a bundle of five lines linked at the top, a dot, a corn glyph (*cintli*)—with position indicating the value of the symbols. We have noted its use in land documents, but it would seem to be more widely applicable and well suited in general for arithmetical calculations.

Documentary Evidence

Two documents utilizing positional line-and-dot notation pertain to the town of Tepetlaoztoc, situated approximately 6 kilometers from the city of Texcoco in the northeastern sector of the Valley of Mexico. We have established that one manuscript, the Códice de Santa María Asunción (housed in the Biblioteca Nacional de México) pertains to the modern Tepetlaoztoc *barrio* of Asunción. The other document, the Codex Vergara in the collection of the Bibliothèque Nationale de Paris, relates to the adjacent barrio of San Gerónimo (*6*). Drawn in the hieroglyphic style of the Acolhua (Texcocan) kingdom, these manuscripts appear to have been drafted circa 1545 (*7, 8*). They report census and cadastral data for 16 small, named localities, and they fit Spanish descriptions of the records maintained by Indian communities in pre-conquest times (*9*).

Each of the two codices is divided into three parts by locality. The first section (*tlacatlacuiloli, tlacanyotl*) contains a census by household, usually five households to a page. The name for each head of household is written in glyphic form beside the conventional symbol for household head (Fig. 1A). The second section (*milcocoli*) consists of a listing of

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land parcels associated with each household (*10–12*). In the milcocoli listing the scribe depicted the approximate shape of each field. The measurement of each side is recorded using lines and dots, a line equal to one linear unit and a dot equal to 20 (Fig. 1B) (*13, 14*). In the Texcocan area, the standard unit of linear measurement was the *quahuitl*, equal to three Spanish varas or 2.5 meters (*15*). In addition to linear measurements around the field perimeter, each field exhibits a glyph in the center that refers to soil type (*16*). Thus, the milcocoli indicates the number of fields possessed by each household, their approximate shape, dimensions, and soil type. The significance of the milcocoli section is that it allows computation of the area of each field and estimation of the amount of land held by tax-paying commoner families (*macehualtin*) in the early post-conquest period (*17*).

The third section of each locality contains another land record labeled *tlahuelmantli*, which follows the same household order as the census and milcocoli (*18, 19*). Since the same household heads for the most part held the same number of properties with corresponding soil types, there is little question that the tlahuelmantli section is a second record of the same lands. However, the two land records differ in several important aspects. The lands in the tlahuelmantli are depicted in stylized form as rectangles of the same size, the majority having a protuberance in the upper right-hand corner (Fig. 1C). In addition to the standardized field shapes, the placement of numbers in the tlahuelmantli is different. The numerical quantities using lines and dots are entered either in the center of the rectangle or on the bottom line, and in the protuberance. When numbers (which never exceed 19) are entered on the bottom line, a cintli glyph occurs near the top border of the rectangle (*20*). In addition, most fields contain a number ranging from 1 to 19 in the upper right-hand corner protuberance. We have determined that these numbers report the area of the field in square quahuitl by use of positional notation.

Positional Notation

The Texcocan positional line-and-dot notation system functions in the following manner. The tlahuelmantli rectangle records numbers in three positions, which we refer to as registers. The first register, located in the upper right protuberance, records the units of 20, indicated by 1 to 19 lines. Groups of 5 are bundled together by a connecting line. The value of this register ranges from 0 to 19, and when the number is zero, the protuberance is either not drawn or is left blank. The bottom line of the rectangle constitutes the second register, and it expresses 1 to 19 units of 20 (numbers 20 to 380). To derive the total value, the number in the second register is multiplied by 20 and then added to the number in the first register. The sum of the two registers never exceeds 399. The central portion of the rectangle consti-

Table 1. Quadrilateral field areas computed from milcocoli dimensions compared with those recorded in the tlahuelmantli. Fields are listed in rank order of discrepancy by size of field. Quadrilateral side lengths are indicated beginning with the left field margin a and proceeding clockwise. Milcocoli areas were calculated by assuming a right angle between sides a and b. Quadrilaterals with quahuitl fractions are not included. Data from Cuauhtepoztla, Códice de Santa María Asunción (*6*)

Household and field number	Milcocoli area (square quahuitls)	Tlahuelmantli area (square quahuitls)	Discrepancy (square quahuitls)	Discrepancy (%)	Side lengths a, b, c, d
		Exact correspondence: N = 9 *(8 percent)*			
HH30-2	97	97	0	0	13, 7, 13, 8
HH4-5	140	140	0	0	14, 10, 14, 10
HH30-4	160	160	0	0	16, 10, 16, 10
HH47-3	180	180	0	0	20, 9, 20, 9
HH64-1	252	252	0	0	14, 18, 14, 18
HH60-1	300	300	0	0	20, 15, 20, 15
HH63-1	346	346	0	0	24, 16, 24, 13
HH31-1	375	375	0	0	15, 25, 15, 25
HH39-7	504	504	0	0	21, 24, 21, 24
		Milcocoli greater than tlaheulmantli: N = 29 *(26 percent)*			
HH29-4	289	288	1	0	30, 10, 31, 9
HH25-1	462	461	1	0	37, 13, 33, 14
HH65-5	160	158	2	1	21, 5, 23, 10
HH16-2	203	200	3	1	25, 8, 26, 8
HH53-3	407	404	3	0	29, 15, 32, 12
HH43-2	529	526	3	0	30, 22, 23, 20
HH26-2	109	103	6	6	12, 7, 12, 12
HH4-8	237	231	6	3	14, 20, 10, 20
HH24-2	127	120	7	6	10, 14, 9, 13
HH44-3	230	223	7	3	15, 14, 17, 15
HH24-3	270	263	7	3	15, 18, 16, 17
HH2-3	347	340	7	2	37, 10, 36, 9
HH17-1	391	380	11	3	19, 13, 37, 24
HH64-2	410	399	11	3	16, 21, 20, 26
HH1-4	899	900	11	1	20, 37, 28, 38
HH26-1	379	365	14	4	20, 18, 20, 20
HH2-5	476	460	16	3	20, 21, 22, 25
HH43-3	180	160	20	11	25, 8, 22, 8
HH38-5	441	420	21	5	22, 17, 23, 23
HH15-2	217	188	29	13	15, 15, 15, 14
HH3-6	184	138	46	25	20, 10, 26, 8
HH58-5	263	207	56	21	29, 8, 27, 10
HH29-5	330	232	98	30	22, 15, 22, 15
HH23-2	309	208	101	33	23, 15, 23, 12
HH41-3	249	132	117	47	19, 15, 15, 15
HH38-7	610	470	140	23	27, 23, 25, 24
HH10-2	551	392	159	29	21, 23, 23, 28
HH2-9	435	252	183	42	24, 17, 26, 18
HH12-1	900	543	357	40	39, 28, 35, 22
		Tlahuelmantli greater than milcocoli: N = 75 *(66 percent)*			
HH59-3	59	60	1	2	8, 8, 7, 8
HH20-3	107	108	1	1	8, 12, 10, 12
HH32-4	152	153	1	0	13, 12, 15, 10
HH41-2	193	194	1	0	19, 10, 18, 11
HH53-2	217	218	1	0	15, 15, 15, 14
HH6-4	359	360	1	0	19, 20, 17, 20
HH21-6	374	375	1	0	30, 13, 30, 12
HH11-3	402	403	1	0	35, 11, 35, 12
HH21-4	409	410	1	0	39, 10, 39, 11
HH55-5	119	121	2	2	12, 11, 9, 12
HH39-9	292	294	2	0	20, 18, 23, 10
HH35-1	406	408	2	0	34, 10, 34, 14
HH5-3	418	420	2	0	20, 20, 20, 22
HH39-5	83	86	3	4	10, 8, 11, 8
HH18-5	157	160	3	2	17, 15, 11, 9

tutes the third register, and expresses quantities of 400 or greater in multiples of 20. Again, to derive the total, the number in the third register is multiplied by 20 and added to the quantity entered into the first register. The Texcocan symbol for 20 (the dot) only occurs in the third register, and according to its positional value means not 20 but 20^2 or 400. The second and third registers never are concurrently used. When there is no entry in the third register, the cintli glyph is drawn toward the top of the rectangle and signifies zero in the third register. Thus, for the first field illustrated in Fig. 1C, the entries are one dot and 11 lines in the third register $[20^2 + (11 \times 20)] = 620$ plus 4 lines in the first register, which totals 624 square quahuitl (*21*). The third field has an area of 333 square quahuitl as recorded by 16 lines in the second register ($16 \times 20 = 320$) plus 13 lines ($\times$ 1) in the first register, and a cintli glyph at the top of the rectangle indicating zero in the third register.

Table 1 (continued).

House-hold and field number	Mil-cocoli area (square qua-huitls)	Tlahuel-mantli area (square qua-huitls)	Dis-crep-ancy (square qua-huitls	Dis-crep-ancy (%)	Side lengths a, b, c, d
	Tlahuelmantli greater than milcocoli: N = *75 (66 percent)*				
HH17-6	185	188	3	2	15, 12, 16, 12
HH50-2	233	236	3	1	20, 10, 21, 13
HH38-3	497	500	3	0	20, 26, 17, 28
HH34-2	180	185	5	3	18, 10, 18, 10
HH39-3	435	440	5	1	20, 20, 24, 20
HH47-2	469	474	5	1	22, 20, 24, 21
HH39-6	118	124	6	5	13, 12, 8, 12
HH53-4	210	217	7	3	15, 14, 15, 14
HH59-5	289	296	7	2	20, 15, 20, 14
HH11-2	200	208	8	4	15, 12, 16, 14
HH16-3	325	333	8	4	38, 8, 39, 9
HH7-3	378	386	8	2	15, 15, 26, 29
HH62-1	229	238	9	4	17, 11, 19, 15
HH15-3	252	261	9	4	14, 16, 16, 18
HH13-4	433	442	9	2	29, 15, 31, 14
HH16-4	169	180	11	7	20, 8, 20, 9
HH24-4	118	130	12	10	10, 10, 12, 12
HH52-5	128	140	12	9	21, 6, 23, 6
HH65-2	307	320	13	4	23, 15, 31, 10
HH52-4	371	385	14	4	22, 15, 22, 19
HH2-2	370	385	15	4	27, 13, 26, 15
HH54-4	472	488	16	3	18, 24, 20, 26
HH54-2	654	670	16	2	37, 16, 38, 19
HH35-2	257	274	17	7	22, 12, 21, 12
HH61-2	318	337	19	6	23, 16, 17, 17
HH45-5	416	435	19	5	40, 9, 35, 14
HH54-3	496	515	19	4	30, 14, 33, 18
HH11-4	288	402	20	9	35, 13, 26, 12
HH3-5	99	120	21	21	11, 9, 11, 9
HH29-2	163	184	21	13	21, 8, 20, 8
HH11-1	246	267	21	9	6, 23, 15, 27
HH17-5	382	405	23	6	17, 23, 15, 25
HH25-2	126	150	24	19	8, 15, 8, 20
HH31-2	363	387	24	7	34, 10, 36, 11
HH54-1	376	403	27	7	20, 18, 22, 18
HH21-5	438	465	27	6	19, 30, 9, 39
HH40-1	466	493	27	6	18, 23, 23, 23
HH22-2	144	172	28	19	12, 12, 12, 12
HH45-1	763	792	29	4	30, 37, 19, 32
HH33-2	400	432	32	8	27, 10, 28, 20
HH44-2	164	197	33	20	11, 16, 20, 8
HH20-1	626	661	35	6	29, 20, 35, 20
HH59-2	174	210	36	21	17, 18, 10, 15
HH1-10	556	593	37	7	35, 16, 37, 15
HH18-4	196	234	38	19	24, 12, 17, 11
HH45-3	232	271	39	17	15, 15, 16, 15
HH51-1	159	200	41	26	23, 9, 23, 5
HH63-2	351	398	47	13	24, 17, 31, 10
HH75-2	262	315	53	20	25, 10, 25, 11
HH56-2	250	310	60	24	20, 17, 15, 14
HH13-2	531	595	64	12	19, 31, 19, 26
HH53-1	966	1034	66	7	66, 14, 68, 15
HH75-5	366	432	66	18	25, 17, 28, 11
HH54-6	549	625	76	14	21, 18, 29, 30
HH75-4	262	360	98	37	25, 11, 25, 10
HH47-1	520	620	100	19	34, 22, 22, 21
HH14-4	287	411	124	43	24, 13, 24, 11
HH18-2	450	591	141	31	39, 23, 15, 19
HH19-3	203	456	253	125	10, 20, 14, 15
HH1-2	504	835	331	66	15, 15, 39, 38

Area Computation

The milcocoli record depicting linear dimensions of individual field boundaries confirms evidence from other native documents and early Spanish descriptions that native Mexicans maintained detailed cadastral records. However, decipherment of the tlahuelmantli adds to this knowledge the hitherto unknown practice of expressing landholdings in terms of area. Thus, it also provides an insight into the practical application of native arithmetic apart from its esoteric use in calendrics and astronomy.

Native methods for deriving area remain to be determined. The milcocoli record does not provide enough information to calculate exact area of fields, particularly the highly complicated forms, since it does not record the true shape by means of angles or auxiliary measures. In other words, the milcocoli cannot be used as a "worksheet" for the tlahuelmantli section. Therefore, there must have been intermediate steps in area calculation between the milcocoli and tlahuelmantli records. A grid system could have been used in the field at the time of survey, or a grid system could have been combined with computation; or the area might have been computed by recording the required information for later computation. The key to discovering the most probable method is in the documents themselves (*22*).

To explore how area was determined and the degree of accuracy achieved by the Texcocans, we calculated the area of quadrilateral fields from Cuauhtepoztla, one of the ten localities in the Códice de Santa María Asunción (*23*). A preliminary analysis indicates that 55 percent of the areas recorded in the tlahuelmantli are within 5 percent of the areas we calculated from milcocoli data; 71 percent of the tlahuelmantli areas are within 10 percent of our values. Therefore, there is considerable agreement between our values and theirs (Table 1). In nine fields, our computed areas and the tlahuelmantli areas correspond exactly; most of these fields were quadrilaterals whose opposite sides were equal. These we assumed to be rectangles for which the length-times-width algorithm applies. Since the Texcocan values correspond to ours, this suggests that they used the same algorithm. The remaining 104 quadrilaterals either have tlahuelmantli areas less than or greater than our estimates. The difference between the two sets of values varies from a negligible 1 square quahuitl to 357 square quahuitls (0 to 125 percent). Tlahuelmantli areas falling below our estimates may be more accurate than ours. Our method assumed one right angle, which would result in an overestimation of area if the assumption did not approximate the actual form of the field. Tlahuelmantli areas greater than our estimates involve other explanations, since in most cases the right angle assumption yields the maximum area possible for the side lengths reported in the milcocoli (*24*).

Some of the discrepancies are explained by two additional algorithms. The tlahuelmantli area of many quadrilaterals that are nearly parallelograms can be obtained by taking the area to be the sum of the areas of two right triangles whose legs are the sides of the fields [for example, the area of HH59-3 (Table 1) was computed as $^1/_2$ (8 × 8) + $^1/_2$ (8 × 7)] (*25*). For more irregularly shaped quadrilaterals, we can frequently derive the tlahuelmantli estimate by summing the area of the largest possible rectangle within the field and the area of the remaining triangles. Thus, the data strongly suggest that Aztec surveyors used some sort of arithmetic calculation to derive area of fields surveyed.

Not all of the differences between our area calculations and the tlahuelmantli values can be accounted for by different methods of computation. Internal evidence in the two codices suggests that the milcocoli and the tlahuelmantli registers were recorded at different times and perhaps by different surveyors. For example, the sequence of fields possessed by each household is not always the same in the two registers. Also, some individual households have added or lost parcels between the two registers. Therefore, it appears that the two registers were either based on separate surveys or that one is an updated modification of the other. If they represent partially or entirely separate surveys, some discrepancies in the linear dimensions of the fields may be due to survey inconsistency. Discrepancies between surveys may also be due to real changes in field size as the result of subdivision or consolidation after inheritance or property exchange; the larger discrepancies probably relate to such realignments.

When all (approximately 1100) of the individual field areas are computed for households or for localities, the discrepancies between our estimates and the tlahuelmantli figures tend to even out. Because some large milcocoli fields are replaced by small fields in the tlahuelmantli and vice versa the areas of the landholdings computed for households remain fairly constant. When areas are computed for localities, the land lost by one household is gained by another, so that the totals show only slight variation (Table 2). Overall this implies not only accurate measuring and careful record-keeping, but also sophisticated methods to compute areas.

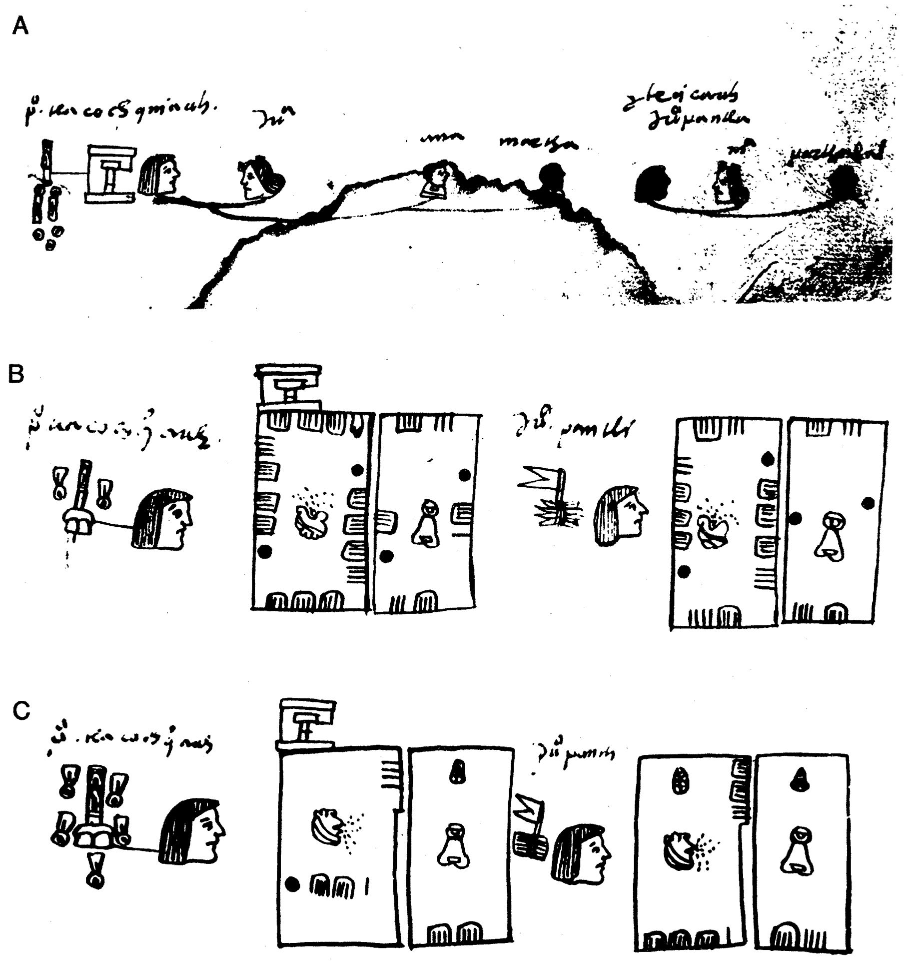

Fig. 1. Portions of the Códice de Santa María Asunción relating to the household of Pedro Tlacochquiauh. (A) The household census, tlacatlacuiloli, shows the head Pedro, his wife Juana, their two young daughters Ana and Martha, and the head's younger brother (teicauh), Juan Pantli, his wife María, and their son Balthasar (Parthasal). The shaded heads indicate that the individual depicted was deceased. (B) The milcocoli section records the approximate shape, perimeter measurements, and soil type of four fields belonging to the household, two to the household head and two to the head's brother. In the numerical notation employed in the milcocoli, lines equal 1 quahuitl and dots equal 20. The hand glyph shown in the first field indicates a fraction of a quahuitl. Quadrilateral fields with quahuitl fractions are omitted from the analysis in Table 1. (C) The tlahuelmantli section shows the same fields as the milcocoli, but they are depicted as abstract rectangles, and a different numerical notation is used. In this positional line-and-dot system, numbers occur in three registers. Lines in the upper right protuberance record units of 20. Lines on the bottom line of the rectangle (second register) or in the center (third register) are multiplied by 20. Dots, which occur only in the third register, equal 20^2 or 400. The corn glyph (*cintli*) at the top margin of the rectangle indicates zero in the third register. The numbers record the area of each field in square quahuitl. [From folios 2r, 10r, and 19v of (*6*)].

Application

The functional significance of calculating land area is most apparent in relation to the system of taxation in ancient Mexico. Landholding commoners paid property taxes (tribute) based on the size and quality of their lands. Hernán Cortés reported that ". . . he who has them can pay tribute because *for each measure* [our italics] so much tribute is charged them according to where the lands are located" (26). Martín Cortés noted that ". . . he who has a piece of land paid a tribute; and the one with two, two; and the one with three, three; and he who had a piece of irrigated land, paid double that of one who had dry land" (27). Since much of the taxed land was in hilly or mountainous terrain, and square or rectangular fields were the exception rather than the rule, the tax system based on standard measures required a mechanism for equating oddly shaped fields with regular ones. This was the task of the surveyor, and as can be seen from the two Tepetlaoztoc codices, even very eccentrically shaped land parcels could be expressed accurately in terms of their square unit content. The importance of survey in Aztec society is reflected in 16th-century Nahua vocabulary, which included numerous terms related to surveying and several referring to incompetent surveyors (28).

There is increasing evidence that the basic area measure of land (analogous to an acre) was 400 square units (20 × 20 linear units). In Tepetlaoztoc, the 400 square unit measure was equivalent to 0.25 hectare [$(20 \text{ quahuitl} \times 2.5 \text{ meters})^2 = 2500$ square meters] (29). In other regions, the length of the linear unit varied, but the concept of 20 by 20 appears to have prevailed. Many extant pictorial documents from Central Mexico depict land parcels as rectangles, with dimensions expressed as multiples of 20. The Tepetlaoztoc codices suggest that some of the fields depicted in other documents might not be actual parcels, but tlahuelmantli-type abstractions. Our data demonstrate that the precise shape of a field was less relevant for tax purposes than the size.

As an example, folios 6 and 7 of the Códice de Otlazpan (30) depict a list of 11 rectangular fields, each 20 *brazas* in width, but increasing in length in regular increments from 10 to 800 brazas (31, 32). They are arranged on the pages according to length, and the tribute to be paid is noted beside each field (33). Of the three types of tribute assessed, two (firewood and turkeys) are fixed quantities regardless of field size; the third (cacao beans or coins) varies in direct proportion to the size of the field. For example, the tribute for a 400 square braza field is 20 cacao beans, indicating a tax rate of one cacao bean per 20 square brazas. The fields listed in the Códice de Otlazpan have been interpreted as actual field dimensions of commoner landholdings in Otlazpan (34). Perfectly rectangular fields seem hardly possible, given the terrain of the Otlazpan area (modern Tepeji del Río), and especially considering that roads, ditches, trees, fences, and constructions make regular shaped fields unlikely even on lacustrine plains. In our view the Otlazpan field list appears to represent standardized abstractions of aggregate field areas with their associated tax levy. As such, the list functioned as a "tax table" from which a person could determine the tax on his holdings. If the Otlazpan tax rate of one cacao bean per 20 square units were applied to Tepetlaoztoc, then taxes on each parcel could also be read directly from the tlahuelmantli second and third registers by reading the numbers at face value rather than as multiples of 20. Thus, a tlahuelmantli field with two dots in the third register would be read as 800 square quahuitl in area and the tax would have been 40 cacao beans (35, 36).

Table 2. Comparison of milcocoli and tlahuelmantli aggregate areas by locality. The first ten localities are from the Códice de Santa María Asunción, and the rest are from the Codex Vergara (6).

Locality	Milcocoli			Tlahuelmantli			Discrepancy
	Area in square quahuitls	Area in hectares	No. of fields	Area in square quahuitls	Area in hectares	No. of fields	Tlahuelmantli − milcocoli (%)
Cuauhtepoztla	130,235	81.4	339	115,990	72.5	229	−12.3
Tecontla	17,715	11.1	42	17,826	11.1	36	+ 0.6
Tlanchiuhca	40,563	25.4	52	46,529	29.1	52	+12.8
Tlantozcoc	21,018	13.1	31	21,636	13.5	31	+ 2.9
Chiauhtenco	16,896	10.6	28	17,241	10.8	28	+ 2.0
Chiauhtlan	26,059	16.3	26	25,313	15.8	26	− 3.0
Cuitlahuac	10,758	6.7	17	12,387	7.7	16	+13.2
Tlaltecahuacan	15,672	9.8	24	16,560	10.4	24	+ 5.4
Conzotlan	11,645	7.3	16	11,035	6.9	16	− 5.5
Zapotlan	10,341	6.5	13	11,061	6.9	13	+ 6.5
Calatlaxoxiuhco	75,400	47.1	209	75,261	47.0	209	+ 0.2
Topotitla	20,544	12.8	38	19,390	12.1	34	+ 6.0
Teocaltitla	54,895	34.3	114	56,352	35.2	114	− 2.6
Patlachiuhca	27,221	17.0	53	26,591	16.6	53	− 2.4
Texcalticpac	110,260	68.9	203	111,879	69.9	199	+ 1.5
Totals	589,222	368.3	1,205	585,051	365.7	1,150	− 0.7

Conclusions

As a result of the decipherment and preliminary analysis of two early postconquest pictorial manuscripts from Tepetlaoztoc, three new features emerge concerning the native arithmetical system of Texcocan-Aztec peoples: (i) positional line-and-dot notation; (ii) a symbol to represent some functions of zero; and (iii) a probable set of algorithms to compute land area. However, since both documents were drafted in the 1540's, the question arises whether the system was pre-Hispanic in use and development or whether it was among the many cultural elements introduced by the Spaniards.

There are a number of features associated with Spanish mathematics that strongly contrast with the Texcocan-Aztec system. Spanish mathematics of the early 16th century was base 10, although some weights and measures were expressed in bases 8 and 12, which derived from earlier conventions in the Mediterranean area. The Arabic system of notation had been adopted in Spain prior to the 16th century, but earlier Roman notation continued to be used extensively until the end of the 17th century. Both Arabic and Roman notations make use of horizontal but not vertical place value. The Texcocan system was base 20, ignored horizontal position, so that the number 23 could be written either as ///· or ·/// (37), and had a positional line-and-dot notation based on vertical position. The zero was highly developed in Arabic arithmetic but only weakly in Texcocan-Aztec. Finally, the Texcocan system was internally consistent throughout and reflected the idiosyncracies of the spoken language. The weight of the evidence, therefore, strongly supports an indigenous development of Texcocan-Aztec arithmetic rather than a post-conquest adaptation to new ideas. The absence of description of the native arithmetical system in the early chronicles suggests that Spanish observers of native culture, so astute in their description of some things, were either ignorant of native arithmetical practice or unimpressed (38).

It is, perhaps, in the practical application of an arithmetical system to derive area that Spanish and native practices most sharply contrast. The Spanish could, but usually did not, express land units in square measures (39). The size of agricultural plots was frequently designated in terms of yield, such as so many *fanegas* of wheat. Also, Spanish land units such as the *caballería* were notoriously variable (40). Traditionally the Spanish were content with imprecise descriptions of land parcels, whereas the native Mexicans were not; this contrast suggests the continuation of pre-Hispanic practices, rather than an adaptation to a Spanish system. As Anderson *et al.* (41) observed ". . . from the mid-sixteenth century, as far back as our selections go, central Mexican Indians were measuring their lands very exactly, down to the yard in both dimensions, using quite sophisticated and individual terminology. At that time Spaniards in Mexico were still transferring land by the league, with no other description than the names of nearby owners or outstanding geographical features."

Three basic concepts of the Texcocan positional line-and-dot system are shared with the Maya: base 20, vertical position, and use of a zero (42). Since Aztec peoples were relatively latecomers to the Valley of Mexico and the zone of high civilization, and since their area of political and economic control eventually extended to the borders of Maya-speaking peoples, the arithmetical system used in the Texcocan area might reflect a direct borrowing from the Maya (43). However, Texcocan-Aztec arithmetic was probably a regional expression of a basic set of conventions and principles known throughout the Mesoamerican area for two millennia or more.

Ciphers expressed by picture symbols were geographically widespread in Mesoamerica, but 16th-century pictorial documents from Central Mexico show that a number of alternate, abstract symbols were also used. In the Otlazpan manuscript, the flag and a bar were alternate symbols for 20. The Codex Kingsborough, also from Tepetlaoztoc, extensively used picture symbols, in many cases combined with lines and dots (35). Although the choice of symbol to record numbers may have been the scribe's, it seems more likely that the determining factor was the object to be counted and that the numerical classifier system of the spoken language was reflected in writing. This could explain why picture symbols appear in one context and abstract symbols in another.

The conventions and principles embodied in the Vergara and Asunción codices are not limited to those documents. The "milcocoli convention" of indicating measurements around the perimeter of fields by lines and dots with values of 1 and 20 is found on the Oztoticpac Lands Map and other Texcocan manuscripts (14, 44). Similarly, the "tlahuelmantli convention" showing abstract fields with area indicated by positional line-and-dot notation has been found on a late 16th-century document from another Texcocan locality (45), and the cintli glyph appears in a context other than tlahuelmantli to clarify place value (46). Since the direct evidence we have for the positional line-and-dot notational system is thus far restricted to the Texcocan province of Acolhuacan and to land documents within this area, it may have been a special system devised for the purpose of land description. It is, however, a system in which arithmetical computations could be accomplished with clarity and facility, in contrast to the more elaborate picture symbols commonly used in tribute records. Until the Tepetlaoztoc documents were deciphered, use of positional notation, the zero symbol, and derivation of land area by Aztec peoples were unsuspected. The significance of the Tepetlaoztoc decipherment is that documents with hitherto undeciphered notations may now be reexamined and our understanding of pre-Hispanic arithmetical practices possibly greatly expanded.

References and Notes

1. D. H. Kelley, *Deciphering the Maya Script* (Univ. of Texas Press, Austin, 1976), p. 3.
2. S. Morley, in *An Introduction to the Study of the Maya Hieroglyphs* (Smithsonian Institution Bureau of American Ethnology, Washington, D.C., 1915), Bulletin 57.
3. Morley commented [(2), p. 91], "By the development of a special character to represent the number 5 the Maya had far surpassed the Aztec in the science of mathematics; indeed, the latter seem to have had but one numerical sign, the dot, and they were obliged to resort to the clumsy makeshift of repeating this in order to represent all numbers above 1. It is clearly seen that such a system of notation has very definite limitations, which must have seriously retarded mathematical progress among the Aztec." Basic misunderstanding of Aztec arithmetic still persists. For example, see M. Weaver, *The Aztecs, Maya, and Their Predecessors* (Seminar Press, New York, 1972), p. 102.
4. H. Cortés, *Cartas y documentos* (Editorial Porrúa, *Biblioteca Porrúa*, vol. 2, Mexico, 1963), p. 73.
5. For a description of Nahua numerical categories see, for example, T. D. Sullivan [*Compendio de la Gramática Náhuatl* (Universidad Nacional Autónoma de México, Instituto de Investigaciones Históricas, Mexico, 1976), pp. 189-195] and M. Orozco y Berra [*Historia antigua y de la conquista de México*, vol. 1 (Editorial Porrúa, *Biblioteca Porrúa*, vol. 17, Mexico, 1960), pp. 443-453].
6. Códice de Santa María Asunción, Apeo y Deslinde de Tierras (de los terrenos) de Santa María de la Asunción, located in the Biblioteca Nacional de México, ms. 1497 bis; Codex Vergara, located in the Bibliothèque Nationale de Paris, ms. Mex. 37-39. A discussion of provenience of these manuscripts is in preparation.
7. C. Gibson, *The Aztecs Under Spanish Rule* (Stanford Univ. Press, Stanford, Calif., 1964).
8. Both codices contain notations in script that were added after the original drafting, including some late 16th-century dates. The documents were signed by Pedro Vasquez de Vergara, who was ordered by colonial authorities to go to Tepetlaoztoc in 1543 to adjust taxes. He was also cited in a legal dispute in the 1550's between the Indians and their *encomendero*, Gonzalo de Salazar. See C. Gibson [(7), pp. 77-80]. The fact that so many individuals depicted on the census are shaded to indicate that they were deceased, together with several dated events on the documents, suggests that the documents were prepared before the onset of the great pestilence of 1545 to 1548.
9. For example, Zorita states that "This principal is responsible for guarding and defending the calpulli lands. He has pictures on which are shown all the parcels, and the boundaries, and where and with whose fields the lots meet, and who cultivates what field, and what each one has. The Indians continually alter these pictures according to the changes worked by time, and they understand perfectly what these pictures show, . . . " in B. Keen, Transl. and Ed., *Life and Labor in Ancient Mexico, the Brief and Summary Relation of the Lords of New Spain by Alonso de Zorita* (Rutgers Univ. Press, New Brunswick, N.J., 1964), p. 110. We know that the Indians of Tepetlaoztoc specifically kept such records because in 1551 the judge assigned to the litigation between the Indians and their encomendero ordered the Indians to exhibit their paintings and explain to him what they meant. (Manuscript, Archivo General de Indias, *Justicia*, leg. 151, ff. 680-750; leg. 157, ff. 41v-45v.) It is quite possible that both codices (or copies thereof) were among the documents presented by the Indians, since the manuscripts had shortly before been certified by Pedro Vasquez de Vergara.
10. E. Seler, in *The Mexican Picture Writings of Alexander von Humboldt* (Smithsonian Institution Bureau of American Ethnology, Washington, D.C., 1904), Bulletin 28.
11. R. Siméon, *Diccionario de la Lengua Náhuatl o Mexicana*, J. Oliva de Coll, Transl. (Siglo Veintiuno, Mexico, 1977).
12. E. Seler [(10), p. 201] translated milcocoli as lands claimed by individuals. J. M. A. Aubin interpreted milcocoli as the "contours ou figure des terres" [in E. Boban, *Documents pour Servir a l'Histoire du Mexique: Catalogue Raisonné de la Collection de M. E. Eugene Goupil* (Leroux, Paris, 1891), vol. 2, p. 15]. Both are accurate descriptions of the content of the milcocoli section. Siméon (11) derived the term milcocoli from *milli*, agricultural field, and *cocolli: cargo, negocio* (charge, business); *riña, disputa, cólera* (quarrel, dispute, anger). Following Siméon's second entry, J. Offner suggests that milcocoli lands are those in dispute ["Law and politics in Aztec Texcoco" (thesis, Department of Anthropology, Yale University, 1979)]; we do not see convincing evidence for this interpretation in the documents, however. In Texcoco, Mexico, certain small rhombus-shaped cakes are called *cocoles*, and if many of these were arranged on a surface, they would create a lattice structure similar to the pattern of milcocoli fields. Perhaps the etymology of milcocoli relates to a Texcocan metaphor for field patterns.
13. H. F. Cline, in *37th International Congress of Americanists: Actas y Memorias* III (Librart, Buenos Aires, 1968), vol. 3, pp. 119-123.
14. ——, *Q. J. Library Congress*, **21** (No. 2), 76 (1966).
15. F. de Alva Ixlilxóchitl, *Obras Históricas* (Universidad Nacional Autónoma de México, Instituto de Investigaciones Históricas, Mexico, 1975), vol. 3, pp. 92-93. "Tenían las casas de longitud que corrían de oriente a poniente, cuatrocientos y once medidas y media, que reducidas a nuestra medida, hacen mil doscientos treinta y cuatro varas y media, y de latitud que es de norte a sur, trescientos veinteséis medidas que hacen novecientos y setenta y ocho varas. . . ." A Spanish vara was equal to 0.84 meter. The problem of equating native, Spanish, and modern linear measurements is discussed by V. M. Castillo F., in *Estudios de Cultura Náhuatl* (Universidad Nacional Autónoma de México, Instituto de Investigaciones Históricas, Mexico, 1972), vol. 10, pp. 195-223; B. J. Williams, *Mexican Pictorial Cadastral Registers: An Analysis of the Códice de Santa María Asunción and the Codex Vergara*, in preparation; see also (13).
16. B. J. Williams, *Geosci. Man* **21**, 51 (1980); in *Actes du XLII Congrès International des Américanistes* (Société des Américanistes, Paris, 1980), vol. 7, pp. 27-37.
17. Since the two documents combined depict more than 1100 land parcels, they provide a record of landholdings unparalleled in the area of Mesoamerica for such an early period. Among other implications, these codices cast doubt upon the "egalitarian" society of the commoners propounded by some writers from the 16th century to the present. Not only did some house-

holds possess significantly greater amounts of land than others, but land of higher soil quality as well. C. Gibson [(7), pp. 268–270] discusses the imbalance in landholdings of the macehualtin and arrays data from various localities. See also B. J. Williams, in *Actes du XLII Congrès International des Américanistes* (Société des Américanistes, Paris, 1980), vol. 9B, pp. 165–175; H. R. Harvey, "Aspects of land tenure in ancient Mexico," in preparation.

18. Papeles de la Embajada Americana, located in the Archivo Histórico, Museo Nacional de Antropología, Mexico.
19. Tlahuelmantli literally means "smoothed, leveled, equalized" according to Siméon (*11*). Not only does the term label land registers in the two codices, but also in another document [(*18*), *3ª* Serie, Leg. 7, Exp. 2-7, Doc. 3], tlahuelmantli appears on two fields drawn in a similar manner and in the same context as in the codices. The literal meaning and the semantic context suggest that "area" is the probable translation.
20. The syllable *çe* or *çi* in name glyphs in the two codices is depicted by a corn glyph, in some cases with and in other cases without husks. Strictly speaking, a rendering without husks would translate as *olotl*, a corn cob without kernels. Even though it is the huskless form that appears in the tlahuelmantli, we translate it as cintli because the glyphic variation apparently was not phonemic in these documents.
21. The area of this field calculated from the milcocoli dimensions in Fig. 1B is approximately 585 square quahuitl (39 × 15). The 39 square quahuitl discrepancy can only be partially explained by the additional fraction of a quahuitl shown by the hand symbol.
22. During fieldwork in the Tepetlaoztoc region, we were unable to recover any vestiges of traditional survey practices, although a number of informants recalled that a rope of standard measure was once kept in the town hall (*presidencia*). At present, tape measures have replaced the standard rope measure. It was also recalled that until recently rope was sold by *mecates* of 2.5 meters, thus reinforcing Ixtlilxóchitl's equation of the standard Texcocan linear measure.
23. We calculated milcocoli areas by two methods. Since fields are not drawn to scale on the manuscripts, angles cannot be directly ascertained. Thus, for quadrilaterals, given the length of sides a, b, c, and d, we assumed a right angle between sides a and b, which allowed calculation of the area of two resultant triangles. For nonquadrilaterals (roughly 40 percent of the cases), field sides were drawn to scale on graph paper and side angles were chosen so as to (i) produce field shapes consistent with shapes observable today, (ii) maximize area, and (iii) preserve the relative shapes recorded in the drawings. Area was then derived by counting squares. Neither method accounted for side lengths in fractions of the standard quahuitl, which, where these occur, results in an understatement of the area figure.
24. In test runs we varied all four quadrilateral angles from 80° to 102° in 2° increments. For 13 of 19 sets of computations, the 90° angle assumption gave areas within 2 square quahuitl of the maximum area generated by varying the angles.
25. Right triangles seem to have been a preferred shape for computation. However, Texcocans were able to calculate the exact area of other triangles as is evident from their area figure of a nearly triangular field depicted in the Códice de Santa María Asunción, f. 16v.
26. *Colección de documentos inéditos . . . de Indias* (Madrid, 1864–1884), vol. 3, p. 542.
27. *Ibid.*, vol. 4, p. 443.
28. B. B. de Lameiras, *Terminolgía agrohidráulica pre-hispánica nahua* (Instituto Nacional de Antropología e História Colección Científica 13, Mexico, 1974).
29. Air photos of Tepetlaoztoc show small relic fields identifiable by soil discoloration in an area where fields recently have been enlarged for mechanized agriculture. These were square fields that measure 400 square quahuitl.
30. Nómina de Tributos de los pueblos Otlazpan y Tepexic, 1549; Códice Mariano Jiménez, located in the Instituto Nacional de Antropología e História, Mexico, 1967.
31. B. Leander, *Códice de Otlazpan* (Instituto Nacional de Antropología e História, Serie Investigaciones 13, Mexico, 1967).
32. In her study of the codex, Leander [(*31*), p. 36] concluded that the Indian braza equaled approximately 2 meters.
33. Folio 6: 20, 40, 60, 80, 100, 200, 300, 400, and 800 brazas in length; folio 7: 10 and 15 brazas in length.
34. Gibson [(7), p. 269] and Leander [(*31*), pp. 79–80] questioned whether the dimensions represented actual field sizes and shapes and suggested that the standard 20-braza width was hypothetical in order to simplify the graphic composition of the document.
35. F. del Paso y Troncoso, Ed. *Códice Kingsborough: Memorial de los Indios de Tepetlaoztoc al Monarca Español contra los Encomenderos del Pueblo* (Hauser y Menet, Madrid, 1912).
36. The daily, weekly, and annual tribute paid in Tepetlaoztoc for the three decades following the Spanish conquest is listed in (*35*). However, this mid-16th-century pictorial manuscript does not indicate the relationship between aggregate tribute and individual land assessments.
37. The notational system employed in the Codex Kingsborough suggests that horizontal position was also important under certain conditions. In this document, 8000, 400, 20, and units are presented in that order. In most but not all instances, picture symbols designate the value of the register, and the quantities are written in Texcocan line-and-dot. An analysis of this document is in progress.
38. Morley (*2*) also noted the lack of Spanish description of the Maya bar-and-dot system.
39. It appears from Spanish textbooks of the period that the average Spaniard possessed only a very basic knowledge of arithmetic and geometry and relied most heavily for solutions to complicated calculations on standardized tables. See, for example, D. E. Smith, *The Sumario Compendioso of Brother Juan Díez: The Earliest Mathematical Work of the New World* (Ginn, Boston, 1921). The original *Sumario* was printed in 1556 in Mexico City.
40. The confusion in Spanish land measurement practices in New Spain resulted in the land ordinances of Marqués de Falces in 1567, reprinted in C. Gibson, Ed., *The Spanish Tradition in America* (Univ. of South Carolina Press, Columbia, 1968), pp. 128–134. These ordinances attempted to standardize procedures. Even had they been carried out to the letter, the results would have been far less accurate than the standard native survey.
41. A. J. O. Anderson, F. Berdan, J. Lockhart, Eds., *Beyond the Codices: The Nahua View of Colonial Mexico* (Univ. of California Press, Berkeley, 1976), p. 5.
42. The functions of the Maya zero are discussed in C. Lizardi Ramos, "El 'cero' Maya y su función," *Estudios de Cultura Maya* (Universidad Nacional Autónoma de México, Mexico, 1962), vol. 2, pp. 343–353, and in C. C. Fulton. "Did the Maya have a zero?" *Carnegie Notes on Middle American Archaeology and Ethnology* (Carnegie Institution of Washington, Washington, D.C., 1948), vol. 3, pp. 233–239.
43. For example, there was some use of the bar-and-dot notation for calendrical dates on Aztec sculpture. See H. B. Nicholson, in *Handbook of Middle American Indians*, R. Wauchope, Ed. (Univ. of Texas Press, Austin, 1971), vol. 10, pp. 92–134. Other groups such as the Zapotec and Mixtec, with whom Aztec peoples were in regular contact, also used the bar and dot.
44. See E. Seler (*10*), pp. 190–196, plates VI and XIII]; see (*18*), Leg. 7-2, Doc. 3 and Leg. 30, Doc. 8.
45. See (*18*), Leg. 10, Exp. 2-29, p. 16.
46. See (*18*), Leg. 30, Exp. 1-35, p. 19.
47. We thank C. Palit, P. Noriega, K. Bennett, D. Price, D. Crowe, and M. Closs for their helpful discussions and reviews of the manuscript, and H. Schlais for assistance in programming and computation. We also thank the Centro Internacional de Mejoramiento de Maiz y Trigo for providing facilities in the field zone. Research was supported by NSF grant BNS 7725659 (to B.J.W.) and a Cyril B. Nave Fund grant (to H.R.H.).