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Mathematics

Saunders Mac Lane

Today, active mathematical research goes on in a bewildering variety of interlocking fields and subfields (*1*). Some of these fields are long-established and relatively concrete ones, like geometry, number theory, or differential equations. Others are new and apparently abstract, such as functional analysis or algebraic *K*-theory. In almost all of these topics, one has today a variety of exciting developments, some closely connected with applications, as in the case of the theory of linear systems. Other active fields, such as algebraic geometry, seem far removed from applications. However, startling connections between topics do appear, as in the current discovery that notions from algebraic geometry can be used in solving nonlinear partial differential equations (the solitons) and in the gauge theories, which are now of central interest in theoretical physics (in relation to instantons and the Yang-Mills equations). Similar surprising interactions occur between apparently very different branches of pure mathematics—as illustrated by the discussion of the Smith conjecture below, or by the examples in *Mathematics Today* (*2*).

Summary. Current research in mathematics involves a wide variety of interlocking ideas, old and new. For example, results about the curves and surfaces defined by polynomial equations, as in algebraic geometry, appear in the study of solitary waves and also in the gauge theories in physics. Centuries-old problems in number theory have been solved, while others have been revealed as insoluble. The classification of all finite simple groups is nearly achieved (and the full treatment will be voluminous); the representation of groups aids in their application to the study of symmetry. These developments, and many others, attest to the vitality of mathematics.

Recent years have seen extraordinary progress in mathematics, as measured either by the large number of famous problems and difficult problems which have been solved (*2, 3*) or by the remarkable number of new concepts which have been developed. I will first give some examples of problems solved and of concepts developed. Then I will try to give more detailed examples of some of the new fields of mathematical research. First I consider some of the old problems recently solved.

Mathematical problems arise naturally, but they can be very recalcitrant, and sometimes literally insoluble. Thus Fourier series have been used to describe periodic phenomena for nearly 200 years. The basic convergence theorem for these series was conjectured about 1910 by the Russian N. Luzin, but was established just in 1966 by Carleson (*4*) (convergence "almost everywhere," given a function in L^2).

In the early 1800's Gauss showed that the complex integers $m + ni$, with $i = \sqrt{-1}$ and m and n whole numbers, could be decomposed uniquely into primes. Gauss also found eight other such cases of unique decomposition for integers $m + n\sqrt{-d}$, where d is positive. It was conjectured that there might be just one more case (making ten in all), but Heegner (*5*), Baker (*6*), and Stark (*7*) proved that there are only nine such cases. A recurring problem is that of determining whether or not such an explicit number is irrational (like $\sqrt{2}$ or $\sqrt[3]{7}$) or transcendental (like e or π). There has been much recent progress here; for example, for the famous Riemann zeta function ζ. R. Apéry succeeded in 1978 in proving that $\zeta(3)$ is irrational (*8*).

The four-color conjecture first appeared 100 years ago; it asserted that on any map of different countries on the globe, it is possible to color the land of each country, using at most four different colors, in such a way that no two countries with a common boundary have the same color. Numerous attempts to prove this failed, but now Appel and Haken (*9*) have established this theorem by using a classical method of attack supplemented by a massive calculation, of about 2000 special configurations, on a computer.

Geometry has made great strides in the classification of closed manifolds. In two dimensions, these manifolds are the closed surfaces such as the sphere, the torus (inner tube), the surface of a pretzel with two holes, three holes, and so on. In fact, this list can be proved to be a complete classification for two-sided sur-

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faces. This classification of surfaces belongs to the subject of topology (the geometry of the simply continuous). A corresponding classification of manifolds in higher dimensions was completely mysterious. Now, by use of a variety of algebraic and geometric techniques (one of them called surgery), considerable progress has been made in getting several stages of such a classification, and there are striking new results for manifolds of dimension 3. However, for manifolds of dimension 4 special difficulties remain (*10*).

Multiplication can be noncommutative, as in the familiar formulas $i^2 = j^2 = -1$, $ij = -ji = k$ for the multiplication of the four basis elements 1, i, j, k of the quaternions. These formulas had been generalized (in the 1930's) to define certain "crossed product" algebras with more basis elements. It was hoped that every suitable algebra (one which is central and simple over a field) could be presented as such a crossed product algebra. Thanks to Amitsur (*11*), we now know that this is false.

Quaternions have inverses, so form a division algebra. The famous theorem that there are no such division algebras with more than four units has recently been illuminated by deep results in topology about the Hopf invariant (*12*).

Geometry, once limited to three dimensions, now extends to spaces with an infinite number of dimensions, such as the Hilbert spaces used for quantum mechanics. Other examples are the infinite dimensional Banach spaces, which arise in the study of properties of functions. All functions f of a given type, each function equipped with its norm $\|f\|$, form what is called a Banach space. It has been hoped that every Banach space had a basis (a suitable infinite set of coordinate axes). We now know that this is not true (*13*), but operators on Banach spaces continue to be very useful in functional analysis.

In algebra, a system of numbers or other things with a well-behaved operation of addition is called an Abelian group—say the Abelian groups of all integers or of all real numbers. An Abelian group which has a basis is said to be free. J. H. C. Whitehead had a plausible conjecture of a good property which ought to characterize these free groups ("every extension with kernel the additive group of integers is split"). Now, thanks to S. Shelah, it turns out that this conjecture can be *both* true and false (*14*). In one model of set theory it is true, in another, false. More generally, other famous problems, such as the continuum hypothesis, have no definite answer or are insoluble.

In the 1960's, kindergarten pupils were taught about sets, as the way to understand the new math. Today we recognize that sets are only one way of formulating a foundation for mathematics, since there is an effective alternative approach by categories (elementary topoi), as shown in (*15*).

Now we return to a consideration of some of the active new branches of mathematics and of a few of the new concepts which have changed our procedures. For example, instead of finding functions to satisfy differential equations, it is now more effective to search for solutions which are "distributions," or generalized functions. The differential operators represented by these equations have been extended to pseudodifferential operators with more flexible properties. The Fourier transform is used systematically to simplify equations; in practical cases it is supplemented by the fast Fourier transform, originally due to Gauss and recently rediscovered (*16*).

When calculus was first discovered, the operation of differentiation was described in terms of infinitesimal real quantities; they were subsequently disavowed as being nonrigorous. This stricture no longer applies. Now there are "nonstandard" models (*17*) of the real numbers which deal with calculus as if there were actual infinitesimals. These models are constructed by techniques of logic; alternative such models can be developed from geometric ideas about "infinitely nearby" points on manifolds (*18*).

These are only samples of the variety of new fields. To deal with this variety, I will try to describe in more detail just a few aspects of mathematical research: the study of symmetry by means of groups, the process of representing one mathematical object (say a group) by another (say a group of transformations), manifolds, the contrast between local and global, the use of these ideas in particle physics, the progress of analysis, and the new understanding of solitary waves. These topics do not pretend to cover all the newer fields of pure mathematics or all the remarkable recent advances in applied mathematics and computer science.

Groups and Symmetry

Symmetry can be analyzed mathematically in terms of groups. For example, an equilateral triangle has six symmetries—three rotations about its center, through angles of 120°, 240°, and 360°, respectively, plus three reflections, one in each of the three altitudes of the triangle. A composite operation, one of these symmetries followed by another one, is also a symmetry, hence one of these six. The six symmetries thus form a group under the operation of composition (multiplication). Here the composition of a symmetry s with a symmetry t is written $s \cdot t$ and means t followed by s.

A larger array of symmetries is exhibited by the icosahedron, a regular-sided solid with 12 vertices and 20 triangular faces, with five faces meeting at each vertex. To count the symmetries, select any one vertex V. Since all the vertices look alike, there is a symmetry carrying V to any one of the 12 vertices V'. Once there, the icosahedron can be rotated about this vertex, so as to carry any one face there into any one of the five faces there; hence, all told there are $12 \times 5 = 60$ rotational symmetries. (They can be suitably represented as all the even permutations of five numbers.) Again, the composite of two symmetries (one following another) is a symmetry, so the rotational symmetries form a group with 60 elements.

More generally, a group G is any set of elements $s, t, \ldots$ which can be multiplied to give products st so as to satisfy all the usual rules for the multiplication of numbers except for the commutative law—this because $s \cdot t$ need not equal $t \cdot s$. When a group G is finite, as in the two examples above, the number of elements in G is called its order. There are also many important infinite groups, such as the group of all rotations about the origin in three-dimensional space. This infinite group is called a Lie group (meaning a continuous group) because each rotation has nearby rotations. A system for classification of all possible simple Lie groups is known—and widely used.

Remarkable progress has been made recently in group theory. We cannot yet describe all possible finite groups, but we are on the verge of discovering all the finite groups which are simple. A group G is called simple if it cannot be collapsed into a smaller group H, mapping each element of G to an element of H, so that products map to products. For example, the group of the triangle is not simple, because it can be collapsed onto a group with just two elements, ± 1, by sending s to -1 if s turns the triangle over, and to $+1$ if it leaves the triangle right side up. (Thus, the reflections in altitudes are the symmetries which turn the triangle over.) On the other hand, one can prove that the group of the icosahedron is simple.

A considerable list of finite simple groups has long been known. For each prime number p, the p rotations of a wheel with p spokes form a simple group. For each integer $n \geq 5$ there is a simple group of order $n!/2$, consisting of all even permutations of n things; for $n = 5$, it is the group of the icosahedron. By adapting some familiar Lie groups from the usual real numbers to appropriate finite number systems, one can construct systematically a list of finite simple groups, said to be of Lie type. But that is not all; beyond these systematic lists there are at least 24 other finite simple groups, called the sporadic groups. One such group of order 7920 was discovered by H. Mathieu in 1861; he soon found four more. No more were found until the last 15 years, when 19 more were suddenly found, beginning in 1965 with Z. Janko's group of 175,560 elements. More are suspected to exist. One of these, called the Fischer-Griess "monster," would have order about $8 \cdot 10^{53}$. More exactly, its order would be the following product of powers of primes:

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

Just this February we had the news (from R. L. Griess, Jr.) that this group has been constructed—by hand calculations rather than with the help of computers, as was the case for some previous sporadic simple groups.

A vigorous group of German, British, and American mathematicians has been working to show that this list of finite simple groups is complete. The proof of this result has been carefully outlined (*19*), and all the partial theorems needed have been proved, with one or two exceptions which are now under attack. When the whole is complete, the total proof will run to almost 10,000 pages of meticulous mathematical argument. This is a proof on a scale never before seen for a single result.

This elaborate determination of all finite simple groups, though a massive job, is likely to be just a starting point for other questions. One is the question of determining explicitly how the simple groups can be put together to make other finite groups. (A part of this is the group extension problem.) The simple groups also have mysterious connections with number theory. There, certain questions, apparently just about whole numbers, can best be tackled by analytic methods, as developed during the 19th century. The study of elliptic functions led to a certain very useful elliptic modular function J expressed in terms of a variable $q = e^{2\pi i\tau}$ as

$$J(q) = \frac{(1 + 240q + \ldots)^3}{q[\prod_1^\infty \ (1 - q^n)^{24}]}$$

This has the power series expansion

$$J(q) = 1/q + 744 + 196{,}884q + 21{,}493{,}760q^2 + \ldots$$

Note the coefficient 196,884. The monster group mentioned above can be described as a group of linear transformations on a vector space of dimension 196,883. The curious fact that this dimension plus 1 is the coefficient of q above is not a coincidence. Mathematicians are now hot on the trail of an explanation of this and related connections between the monster group and number theory.

Representation of Groups

The deeper understanding of the structure of groups often depends on their representation by transformations. This refers to one-to-one transformations of a vector space into itself, say by rotation or reflection of the space. Each such transformation on an n-dimensional space can be expressed as an $n \times n$ matrix of numbers (real numbers, complex numbers, also numbers from a finite number system). Thus a representation of a group G is a process which replaces each element s of G by an $n \times n$ matrix S (or the corresponding transformation) in such a way that the product st of group elements is represented by the product matrix ST. It becomes important to break any one such representation down into simpler ones (by breaking each $n \times n$ matrices into smaller square blocks). The simplest representations, those that cannot be further broken down, are called irreducible. Each such irreducible representation is summarized by its character, which is the function giving for each group element the sum of the diagonal terms in the representing matrix. Such representations and characters, for all the various number systems, can be determined for all finite groups, and play an essential role in the description of special groups such as the monster discussed above.

Infinite groups may have a continuous structure which can be "bounded" (technically, compact Lie groups). They also may be analyzed by representations, especially by rotations (that is, unitary transformations), again using characters. In many important cases, the geometry involved can be described purely algebraically, in terms of a so-called Lie algebra. Much has been (and will be) discovered about the structure of these algebras. Even "bigger" infinite groups, like the groups of all rigid motions (including translations) or the Poincaré group of relativity theory, have important representations which have been analyzed, starting with the work of E. Wigner in 1939. These groups require representations in infinite dimensional spaces (Hilbert spaces) and appear in quantum mechanics and elsewhere in physics (the "eightfold way").

More recently, Harish-Chandra has made a study of infinite-dimensional representations of arbitrary "semisimple" Lie groups in terms of traces (generalized characters) and distributions (generalized functions). This work is strongly connected to problems in analysis, in a subtle generalization of the way in which the ordinary trigonometric functions provide representations of the (Lie) group of all rotations θ of an ordinary circle; these representations send each angle of rotation θ to $\cos 2n\theta + i \sin 2n\theta = e^{2in\theta}$, just as in the Fourier analysis used for the study of periodic functions. In a long series of papers, Harish-Chandra (*20*) has completed this harmonic analysis, in a theorem involving "invariant eigendistributions," as represented by suitable characters, and providing, as in the case of finite groups, a suitable decomposition into irreducible characters. There are profound connections with number theory ("automorphic forms") which have been explored in the last decade by R. P. Langlands and others (*20*). Work on these connections is likely to continue well into the next century.

Manifolds and Topology

Continuous geometry (that is, topology) has made and will make extensive progress with subtle algebraic methods and deep results about manifolds. A topological manifold can be described as pieces of Euclidean space glued together (for example, a torus is a patched inner tube). When the pieces are glued together by linear maps, the result is called a piecewise linear manifold. Similarly, differentiable gluing gives a differentiable manifold. For example, an ordinary sphere can be several lunes (or two overlapping hemispheres) glued together; both give the same differentiable structure on this (two-dimensional) sphere.

A famous and long unsettled conjecture, the Hauptvermutung, claimed that any piecewise linear manifold must have an essentially unique triangulation (a linear gluing). Quite recently, D. Sullivan proved this for a large class of manifolds. Kirby and Siebenmann (*21*) proved that most topological manifolds actually can be triangulated (*22*), but they also found examples with no such triangulation. Differentiable manifolds became of central interest when J. W. Milnor discovered the surprising fact that the ordinary seven-dimensional sphere had more than one such differentiable structure.

The classification of manifolds originally used simple qualitative properties of manifolds as measured by numbers, such as the number of connected pieces. Now the measurements of connectivity are made by much more subtle algebraic invariants, capable of making much finer distinctions, and giving invariance under deformation (or homotopy).

The new ideas of category theory were invented to describe this process of mapping one field, such as topology, to another, such as algebra, and category theory now has developed into an interesting subject on its own.

Perhaps the most striking aspect of topology is the effective way in which geometric and algebraic techniques have been used first to translate classification problems in manifold theory to problems in homotopy theory and then to solve the latter by use of the invariants of algebraic topology.

The procedures involved illustrate a fundamental feature common to a great deal of modern mathematics, from algebraic geometry to functional analysis. Time and again, motivated by specific concrete problems, mathematicians have been led to invent far-reaching abstract theories of great depth and beauty and with new applications far removed from the original motivating problems. In the interests of brevity and accessibility, I can pay little attention in this article to many of the most central and mature areas of mathematics, precisely because their maturation has involved such powerful, and not easily summarized, abstract ideas and techniques.

Local and Global Geometry

Differential geometry is the branch of mathematics in which calculus is used to get an understanding of geometric properties (curvature, and so on) of curves and surfaces. Much recent progress in differential geometry has depended on an interaction between local properties and global properties. A local property of a surface is one which can be calculated at a point of the surface by using only properties of the neighborhood of that point, as expressed, for example, by coordinates valid near that point. The (Gaussian) curvature of a surface at a point is one such local property: it can be measured in terms of the angle sums of small triangles near that point; it has the remarkable property that it depends only on angles and distances measured in the surface itself, and not at all on the way in which the surface is embedded in its ambient space.

The Euler characteristic of a polygon is an example of a global property. Let V be the number of vertices, E the number of edges, and F the number of faces of the polyhedron; thus for a tetrahedron $V = 4$, $E = 6$, and $F = 4$, while for a cube $V = 8, E = 12$, and $F = 6$. In both cases $V - E + F = 2$; the same equation holds for an octahedron or for any polygonal subdivision (regular or not) of a figure like the surface of a sphere. For a different closed surface such as a torus, any subdivision of the surface into triangles or polygons gives a different sum, $V - E + F = 0$, with the same answer no matter how the subdivision is made. For any surface S the corresponding quantity $V - E + F = \chi(S)$ is an integer depending only on the surface S and not on the way the surface is subdivided into vertices, edges, and faces; $\chi(S)$ is called the Euler characteristic of the surface. It is a global property of the surface, since it depends on the way the whole surface is fitted together.

A typical theorem of differential geometry is the Gauss-Bonnet theorem, which asserts that the Gaussian curvature of the surface, when integrated over the whole surface S, always gives the answer $2\pi\chi(S)$. This is a typical relation between a local invariant (the curvature at each point) and a global one (the Euler characteristic). Recent years have seen a much better understanding of the proof of this theorem, its generalization, and its consequences. One simple consequence is that "you can't comb the hair on a billiard ball, but you can on a torus." Specifically, it is easily possible to lay down at every point of the torus a nonzero tangent vector to the surface, varying continually with the point, but this cannot be done on a sphere, where any such field of nonzero tangent vectors must have at least one singularity (see the whorl of hair at the back of a typical human head). Much subtle work in differential geometry and topology has been devoted to extending these results to the possibilities of other vector fields on other surfaces and on manifolds of higher dimensions. For example, in 1944 and 1945, Chern (*23*) extended the Gauss-Bonnet theorem to Riemannian manifolds of arbitrary dimension and subsequently developed the resulting characteristic classes [like $\chi(S)$] as global invariants of these manifolds. This now has repercussions in the study of ways of subdividing a manifold up into leaves (a foliation).

Geodesics provide another striking example of the interrelation of local and global properties. A curve C on a surface is geodesic if it gives shortest distances; that is, if the path along C from a point p on C to another point q on C is a shortest possible path on the surface from p to q, provided p and q are not too far apart. Thus, on an ordinary sphere, each great circle is a geodesic. Geometers had exhibited explicit geodesics on many special surfaces. The natural desire to get a more general understanding led, through the work of Poincaré, Birkhoff, Morse, Lusternik-Schnirelmann, and others to a general existence theorem, which states that on a given closed convex surface there always exist at least three distinct and simple closed geodesics. An example due to Morse had already indicated that three is the best possible number in such a result. Klingenberg (*24*) and others have extended such results from surfaces to higher dimensions.

These problems about geodesics involve finding a curve of minimum length. Other such minimum problems have been codified in the calculus of variations ("vary the curve so as to make sure you have the minimum length"). Thus for a function $f(x_1, \ldots, x_n)$ one knows, as in the calculus, that at a minimum of f all the partial derivatives $\partial f/\partial x_i$ will vanish; the same happens at a maximum. More generally, a point $(x_1, \ldots, x_n)$ where all these first partial derivatives vanish is called a critical point of the function f; it can be a minimum of f, a maximum of f, or a saddle point, as in the familiar case of the point at the top of a mountain pass, when $f = f(x,y)$ measures height above sea level. For a function of n variables there are $n + 1$ such types of critical points. When the numbers of critical points of each type for a given function are counted up, it turns out that they must satisfy a number of special relations, as discovered (in the 1920's) by Marston Morse.

The resulting Morse theory for critical

points has played an essential role both in the construction of geodesics (critical points for the function giving the length of a curve) and in the classification of higher dimensional manifolds, using the critical points of functions which can be defined on the manifold (*25*).

When a soap film is spanned in a loop of wire the film forms a surface with the minimal possible area (for that given boundary loop). Such a surface is called a minimal surface; as a surface of minimum area it is a two-dimensional analog of a geodesic (a curve of minimum length); there are higher dimensional analogs. The Plateau problem required a proof of the theorem that there always exists a minimal surface for a given boundary loop of any shape, provided only that it is rectifiable (has a length). This central theorem was first proved in the 1930's by Jesse Douglas and Tibor Radó. Their solutions, however, did not exclude surfaces which might have singularities which are branch points. This possibility has been eliminated in a more recent study by Osserman (*26*), who showed that the branch points are not present.

A minimal surface can also be described as a solution of a certain (nonlinear) partial differential equation (PDE, for short). The study of PDE's is a vast subject boasting considerable recent progress. The subject uses many recent and subtle existence theorems, showing that suitable PDE's do have solutions with appropriate boundary values. There is also a notable nonexistence theorem, due to Lewy (*27*). He exhibited a linear PDE in three independent variables which had smooth coefficients but no smooth solutions whatever—the trick being that the constant term was smooth (had derivatives of all orders) but was not analytic. This striking example has stimulated much further research, by L. Hörmander and others, on existence theorems for PDE.

Some questions of this type involve complex geometry; this is always geometry in an even number of (real) dimensions, because each complex number $z = x + iy$ involves two real coordinates x and y. More generally, consider $2n$-dimensional space C^n, with coordinates given by n complex numbers, and let D_1 and D_2 be well-behaved sets in the space (technically, smooth and strictly pseudoconvex domains). A fundamental problem is that of determining whether D_1 and D_2 are equivalent in the appropriate sense, through a function F mapping D_1 onto D_2, with this function F and its inverse both well behaved (that is, complex analytic). Crucial information on this question depends on how well F behaves as one approaches the boundary of D_1. Vormeer showed that F is necessarily continuous up to the boundary, and then Fefferman (*28*) showed that it is smooth all the way up to the boundary. Fefferman used the Bergman kernel function K_D for the domain, depending on two variable points in the domain. This result, in turn, could be used to get information about the solution of certain Monge-Ampere PDE's which arise in geometry.

Geometry and Elementary Particle Physics

One of the exciting developments in high-energy physics over the past decade has been the gradual emergence of theories which have brought some kind of order into the realm of elementary particles. It is now widely believed that what are called gauge theories will provide a satisfactory explanation for the experimentally observed phenomena, including the diversity of fundamental particles such as the electron, proton, neutron, and mesons.

Gauge theories are mathematically sophisticated and involve a number of key ideas which mathematicians have worked on, quite independently, for many decades. As a result, these developments have, in the past few years, led to an increasing interaction between mathematicians and theoretical physicists. Not since the days of Einstein's work on general relativity some 60 years ago has there been such a ferment of activity on the frontier between mathematics and physics. In fact, general relativity, which interprets the gravitational field as the curvature of space-time, is itself a gauge theory, and so is Maxwell's theory of electromagnetism. In Maxwell's theory the electromagnetic field can also be viewed as curvature, not the curvature of space-time but the curvature of some fictitious phase space. To see this, imagine that at each point of space-time there sits a circular dial marked off in degrees. In order to standardize these dials, imagine an observer moving from one point to another, taking his dial with him. The geometric interpretation of Maxwell's theory says that if the observer took a different route his dial would show a different reading at the end, the deviation depending on the strength of the electromagnetic field in the region traversed.

In modern gauge theories, other forces involved in the nucleus are interpreted in a similar way as causing curvature or distortion. However, the angular phase as measured by a dial is now replaced by a multidimensional phase described not by a plane rotation but by a rotation in space of three or more dimensions. Because rotations in higher dimensions do not commute (that is, the results depend on the order in which rotations are performed) the mathematics is much more difficult; in particular, the field equations become nonlinear (differential equations). Solving these equations is now a serious problem. Some surprising successes have been achieved in this direction by the use of modern mathematical techniques coming from unexpected areas such as algebraic geometry.

Topological ideas have been found to play a very significant role in these nonlinear theories. This can be understood naïvely in the following way. Linear equations are in principle easy to solve, while nonlinear equations can be solved approximately by comparing them with linear equations just as a curve can be approximated by a tangent line. However, such approximations are inherently local—that is, work well only in a small region. Topological arguments are a very valuable way of obtaining global information from local data.

At present, it is clear that many geometric ideas and techniques will be involved in the further development of gauge theories (*29*). One outstanding problem that remains is to understand the quantum theory of gauge fields. This is essential to the physics because one is dealing with phenomena at very short distances (or equivalently, at very high energies), and the mathematical apparatus to deal satisfactorily with quantized gauge fields is still in its infancy. It remains a challenging problem for the future and one which will probably require the joint forces of mathematicians and physicists.

Solitary Waves

An example of the prospects in applied mathematics is the rapid progress in the study of solitary waves. Such waves were first noted about 1840 by Scott Russell in the Edinburgh-Glasgow canal, where he chased a single long-crested wave moving without change of form for a mile and a half down the canal. At first it was not known how to obtain such wave forms from the standard partial differential equations for surface water waves. A suitable model equation was found by several authors, including D. J. Korteweg and G. de Vries in 1895. If u is the height of the wave above the standard water level in the canal at time t and at distance x down the canal (from the start of the disturbance), then u as a function of x and t satisfies the Korteweg–de Vries partial differential equation

$$u_t + u_x + uu_x + u_{xxx} = 0$$

where u_t indicates the partial derivative of u with respect to t and u_x that with respect to x. This model equation can be explained as consisting of a basic term ($u_t + u_x = 0$) expressing the propagation of the wave form in one direction, plus the nonlinear effect uu_x and the effect of dispersion given by u_{xxx}. It applies, to a good approximation, to waves of small amplitude and correspondingly long wavelength. The equation has explicit traveling wave solutions, one for each possible amplitude, expressed in terms of the hyperbolic secant. These special solutions provide an approximate description of the solitary waves as observed by Scott Russell. Moreover, the equation can be used to demonstrate what happens when two solitary waves of different amplitudes meet: the wave of larger amplitude catches up to the smaller wave; there is a period of apparently confused interaction, and then the larger amplitude wave emerges ahead, its form unchanged, followed by the smaller wave, equally unaffected by the collision.

The same Korteweg–de Vries equation applies to many other kinds of physical phenomena (*30*): to magnetohydrodynamics in a cold collision-free plasma (C. S. Gardner and H. Morikawa), to longitudinal waves in a one-dimensional lattice of equal masses coupled by nonlinear springs, to pressure waves in bubbly liquids, to waves in elastic rods, and to at least 20 other such phenomena. Given this variety of applications, more incisive methods of solution were needed. Active work led to several surprises. First, the scattering methods used in treating equations such as the Schrödinger equation of quantum mechanics could be applied—backward. These scattering methods depend on finding suitable scattering data such as eigenvalues, the continuous spectrum, and spectral densities. For the Korteweg–de Vries equation, one uses inverse scattering: starting from simple laws giving the time evolution of the scattering data, one recovers the necessary facts about the solutions u of the original equation. It is by using the inverse scattering method that one can demonstrate mathematically the interaction of two different solitary waves (solitons) as noted above.

Next, the usual physical invariants for such equations—invariants such as energy and momentum—apply here also, but for this differential equation it turns out that there is an infinite number of different such invariants. Finally, and even more recently, there have turned up several connections between the Korteweg–de Vries equation and properties of curves defined by algebraic equations (more exactly, such curves with a few points omitted). Curves such as these—and their manifold arithmetic and algebraic properties—had long been studied in pure geometry; what is surprising is the connection with the behavior of water waves. This is a topic requiring more elucidation, as a part of the current active and extensive research on the Korteweg–de Vries equation. Some related recent research has used the powerful techniques of bifurcation theory to provide solitary wave solutions for the exact differential equations for surface waves.

Interaction of Disparate Fields

Recent progress in mathematics depended to a remarkable extent on the interactions between different and apparently quite remote subfields. One example is the appearance of ideas from "pure" algebraic geometry in applied questions about gauge theories and about the Korteweg–de Vries equation, as noted above. Another striking example concerns the recent proof of the Smith conjecture about periodic transformations on a three-dimensional sphere.

If the ordinary sphere S^2 is rotated by $360°/n$ about its north-south axis, the resulting transformation T, repeated n times, is just the identity ($T^n = 1$); one thus says that T is periodic of period n. The only points left fixed by T itself are two points, the north and south poles; these two points may be said to constitute a zero-dimensional sphere S^0.

Now consider a three-dimensional sphere S^3, given as the locus of $x^2 + y^2 + z^2 + t^2 = 1$. A transformation T of S^3 into itself is called periodic if T, repeated n times for some integer n, brings every point back to itself ($T^n = 1$). For example, a rotation in three space (rotation about the z axis in the x, y, x coordinates, leaving t fixed) through $360°/n$ will be periodic; this time all the points on the circle $z^2 + t^2 = 1$ are left fixed. More generally, it is true that any differentiable periodic transformation T on this three sphere, even if it is much more irregular, will leave fixed a number of points forming a simple closed curve (S^1, the continuous image of a circle). However, this curve of fixed points might be knotted (say, as in an ordinary overhand knot). Thirty or more years ago, P. F. Smith conjectured that this curve would never be knotted. In 1979 this conjecture was verified—thanks to the joint efforts of a half-dozen mathematicians using different special techniques, ranging from minimal surfaces and the properties of three-dimensional spaces which are non-Euclidean in a hyperbolic way to the properties of the K-groups, a recently developed algebraic structure attached to topological spaces.

We note a few other examples of interactions between different fields. Calculus of variations, a venerable subject as used for geodesics, minimal surfaces, and the like, has had a rebirth and new development in the theory of optimal control. Precise ideas about which functions are actually computable have been developed in the last 40 years in symbolic logic following ideas of K. Gödel; they have found use in number theory, where they provide a proof that there are equations in whole numbers (Diophantine equations) for which there is no systematic method of solution. These ideas are also used in computer science, to measure complexity of a computation. Algebraic geometers spent 25 years to achieve the solution (*31*) of a conjecture of A. Weil about "rational" points on certain algebraic curves. The result turned out to give also a solution to an old problem of Ramanujan. He considered a certain useful function $\tau(n)$; it can be described as the coefficient of q^n which appears in the expansion of the quantity $q\Pi(1 - q^n)^{24}$ that comes up in the elliptic modular function mentioned above. He asked for estimates of the size of this quantity, especially in the case when n is a prime number $n = p$. It was relatively easy to show that $\tau(p)$ is at most $2p^6$; the surprising (and much harder) new result (*31*) is that $\tau(p)$ is at most $2p^{11/2}$.

New results in the current literature are summarized each year in two volumes of *Mathematical Reviews*, which ran in 1979 to more than 40,000 articles subdivided into 61 specialties, ranging from mathematical economics and optics to several complex variables and finite difference equations. Within this range of topics, abstract harmonic analysis (of periodic functions) connects with problems of probability theory, while the very general concepts of category theory are of assistance in the study of finite state automata. Mathematics presents an intricate, tightly woven net of precise, detailed structures and techniques for the analytic understanding of the phenomena of the world.

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Physics

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Those aspects of man's knowledge of the natural universe that are both universal and permanent make up physics: universal in the sense that behavior unraveled in an earthbound laboratory is assumed to be equally valid in the farthest reaches of the universe, and permanent in the sense that, once demonstrated conclusively now, an addition to our understanding of nature is assumed to be valid back to the beginning of the universe and forward into the far distant future. Although physics is frequently among the most arcane of the sciences, its applications have profoundly affected the lives of every human. Yesterday's frontier has very often proved to be tomorrow's application.

The past 5 years have been the most exciting, challenging, and productive in world physics since the late 1920's, just after the discovery of quantum mechanics. We appear to be on the threshold of an entirely new understanding of physical phenomena on a more fundamental level than ever before. And, gratifyingly, we find that in contrast to the fragmentation that characterized physics—and indeed many other sciences—in past decades, the underlying unity and coherence of our science is once again emerging. Concepts and techniques developed in one subfield are rapidly exploited in others, to the enrichment of all.

Obviously, this brief review of so dynamic a field cannot possibly be complete. What I have tried instead to do is select, in each of the major subfields of physics, a few areas which will illustrate the renaissance and in which I believe major progress will be made in the years ahead. I shall follow the natural hierarchy, from the microscopic to the macroscopic, from the depths of the atomic nucleus to the fringes of the known universe.

Elementary Particle Physics

Concerned with the ultimate structure of matter and the fundamental forces and symmetries of nature, elementary particle physics moves inexorably to ever higher energies to probe ever shorter distances and to establish the ordering principles that nature has chosen.

The annus mirabilis in this field was 1974. Not only was there evidence produced for an entirely new building block of nature—in the discovery of the ψ particles—but also the grand unification theories aimed at bringing together electromagnetism and both the weak nuclear force (responsible for radioactivity) and the strong nuclear force (responsible for binding nuclei and for nuclear energies) were first introduced.

Prior to 1974, after enormous effort, both experimental and theoretical physicists generally had concluded that three massive, fractionally charged entities, the quarks, were the building blocks underlying the natural universe. The discovery of the ψ particles showed conclusively that there was yet a fourth quark, carrying a totally new quantum attribute arbitrarily christened as "charm"; within the past 2 years the discovery—at even higher energy—of the Υ particles proved the existence of a fifth quark, this one with a quantum attribute equally arbitrarily called "bottom" (see Fig. 1). And almost all physicists are convinced, on the basis of theoretical symmetry arguments, that a sixth quark carrying the attribute "top" will be discovered as somewhat higher energies become available to experimenters. The necessary facilities are now under construction, both

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