Phase Changes: A Universal Theory of Critical Phenomena

In the last 2 years a sweeping new theory of critical phenomena has been proposed. Although primarily a result of new physical insight, the advance involves group theory in some forms of its expression.

Everyone knows that when water boils at 100°C, a dramatic change in the density of the liquid takes place as it turns to gas. But when water boils at higher temperatures, as it must if the pressure is greater than atmospheric pressure, the change of density that occurs in the liquid-to-gas transition decreases until at a sharply defined critical temperature it disappears entirely. Above the temperature of 374°C, water exists only as steam. All liquid-gas systems behave in a similar way, though the critical temperatures vary widely from one substance to another.

But liquid-gas systems are examples of only one type of critical phenomenon. Liquid solutions, biopolymers, superfluids, liquid crystals, alloys, superconductors, and ferromagnetic metals all undergo phase transitions that are also classed as critical phenomena. If a nickel magnet is heated, its magnetization decreases until it abruptly vanishes at 354°C, the critical temperature. At room temperature the crystal structure of a brass alloy composed of equal numbers of copper and zinc atoms is very regular, with atoms of the two metals located at alternate sites in a cubic lattice. But when the alloy is heated, the pattern becomes less regular, as measured by x-ray or neutron scattering, until it is completely destroyed at 466°C. Yet another example of a critical phase transition occurs in liquid helium. Below the critical temperature of −271.0°C, helium can exist in a superfluid phase. But above that temperature the superfluid, which has many unusual properties, disappears.

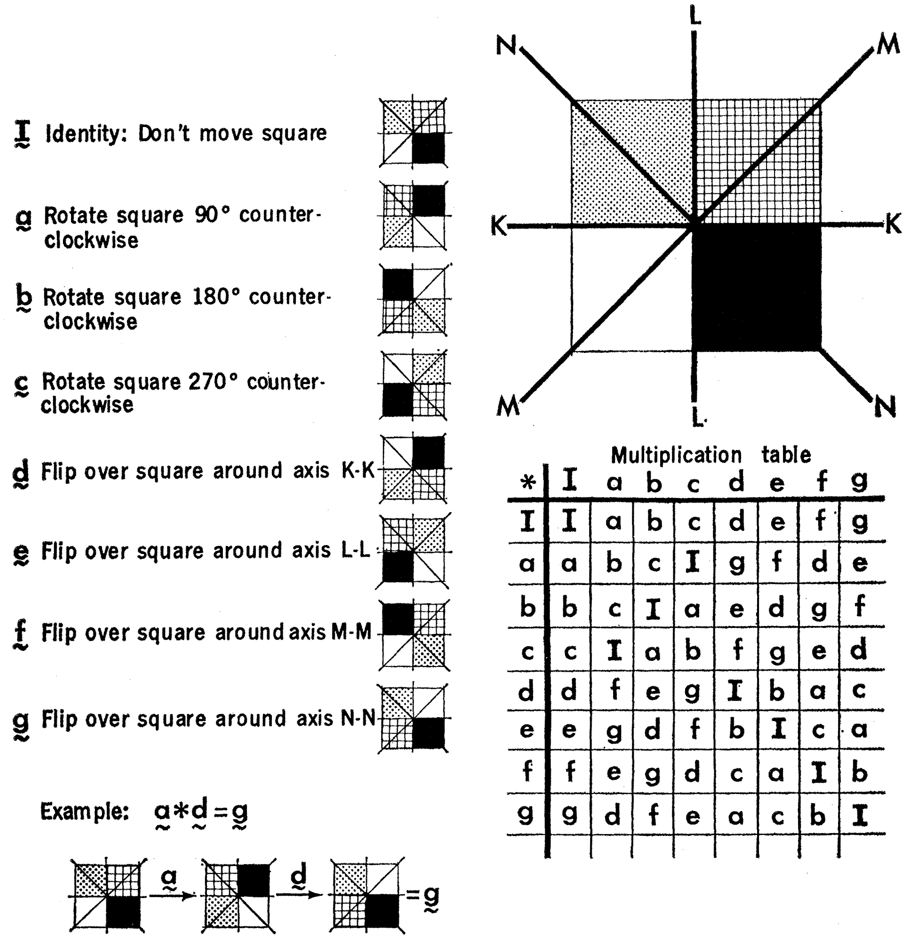
By about 1965 it was clear that the classical theories of phase transitions were inadequate to describe critical phenomena, and in the following years many theorists began to realize that different classes of critical behavior are related in ways that are essentially independent of the physical details of the different systems. While the critical temperatures of various systems depend on specific physical details, such as the strength of a molecular force or the interatomic spacing in a lattice, the qualitative aspects of critical behavior are apparently independent of those details and constant, not only within classes of critical phenomena, but also from one class to another.

For example, the difference in density between liquid and gas—for water or any other liquid—decreases as the temperature approaches the critical temperature (T_c) with the particular functional dependence $(T - T_c)^{\beta}$. The exponent β has been measured to be very nearly ⅓. The magnetization of nickel and the degree of order in brass approach zero with the same qualitative dependence on temperature, and again β is nearly ⅓. Furthermore, β is only one of the so-called critical exponents that seem to have the same values for many different classes of critical phenomena. Besides an exponent to describe the coexistence of different components, the complete

(Continued on page 149)

table such as that given in Fig. 1. Often it is convenient to think of groups not as composed of particular objects such as numbers or rotations, but as a set of abstract elements subject to the combination rule expressed by a multiplication table. In this more general view, groups with the same number of elements and the same multiplication table are essentially identical. The enumeration of all abstract groups—which have been compared to the grin that remains when the Cheshire cat fades away—and the determination of their properties are the basic tasks of group theory.

—A.L.H.



Multiplication table

*	I	a	b	c	d	e	f	g
I	I	a	b	c	d	e	f	g
a	a	b	c	I	g	f	d	e
b	b	c	I	a	e	d	g	f
c	c	I	a	b	f	g	e	d
d	d	f	e	g	I	b	a	c
e	e	g	d	f	b	I	c	a
f	f	e	g	d	c	a	I	b
g	g	d	f	e	a	c	b	I

Fig. 1. A finite group with eight elements. [Kenneth Smith]

description of the thermodynamic properties of a system nearing the critical point requires a critical exponent (α) to describe the rate of divergence of the specific heat, a critical exponent (γ) associated with the divergence of the compressibility or magnetic susceptibility, and a critical exponent (δ) related to the critical isotherm.

A number of semiphenomenological theories were developed in the mid-1960's to explain the apparent universality of the critical exponents—that is, for example, that β is apparently the same for many classes of three-dimensional systems—and to specify the values of the exponents. Benjamin Widom at Cornell University, Ithaca, New York, and Leo Kadanoff at Brown University, Providence, Rhode Island, were among those who produced essentially equivalent theories. These theories gave relations among the critical indexes, but did not predict actual values for them.

In 1971 Kenneth Wilson, at Cornell, dramatically changed the status of theories of critical phenomena by presenting for the first time an analytic approach to the calculation of the critical exponents of three-dimensional systems. Wilson's theory is based on the suggestion that if you change the scale of length when observing critical phenomena, the problem is not altered in any essential way. The operations involved in rescaling the length can be described by a group, which Wilson calls the "renormalization group." It is technically only a semigroup, because it has no inverse.

A magnetic phase transition is a good example of how critical phenomena change when the scale of length is changed. The relevant parameters are temperature, magnetic field, and the correlation of adjacent magnetic domains. If the scale of length (the number of atoms per unit length in the crystal) were changed, the values of the other parameters would also change. But suppose you had two experimenters measuring the various parameters of the magnet, including length, with meters that were calibrated differently. Suppose magnetic energy were measured in units of temperature (kT), where k is the Boltzmann constant, and temperature were measured as the percentage deviation from the critical temperature $(T - T_c)/T_c$. Full scale on one experimenter's temperature meter might represent 1 percent deviation from criticality, while full scale on the other experimenter's meter might represent 1/10 percent deviation from criticality. Nevertheless, as long as temperatures were near the critical point, the recalibration of the other meters could be arranged so that the results seen by the two experimenters (if length is changed) will appear to be identical. The situation just described is, of course, not physically realistic because the intermolecular distances in a crystal are not parameters that can be freely altered. However, very near a critical point, phenomena become more and more independent of atomic length and energy scales. Hence the universality of critical phenomena.

The mathematical operation of renormalization is in effect the recalibration of the meters. The group property of closure is satisfied by the renormalization group because if you made two suitable recalibrations of the meters the result would be equivalent to another recalibration. Because the scale of the length meter could be changed by any amount, as long as the scales of the other meters were properly coordinated, the renormalization group is a continuous group. (Essentially all infinite groups that describe topological transformations are continuous.)

Predominance of Geometry

Wilson not only established detailed reasons for believing that the ideas of renormalization and universality were appropriate for critical phenomena. He also developed a theory which predicts that the values of critical exponents depend only on a very general symmetry property related to the rotations and reflections of atomic spin (which is also a group property), and on the number of spatial dimensions of the physical system. (A thin film is a system with two dimensions rather than three.) Critical phenomena appear to depend much more on geometry than on physical detail.

The equations developed by Wilson from his renormalization group approach could predict all the critical exponents for systems of any spin symmetry in any dimension, but they are extremely difficult to solve. In 1972 Wilson and Michael Fisher, also at Cornell, reported another major advance, which for the purpose of making practical calculations is just as important as the first. Noting that the classical theory of phase transitions predicts a particularly simple form for systems of more than four dimensions—for instance, the critical exponent γ becomes 1—Wilson and Fisher developed a calculation starting from a simple solution in four dimensions to find values for the critical exponents in the more realistic situations with three, two, and one dimensions. Specifically, they performed a perturbation around four dimensions in order to solve the equations by expressing the dimensionality as $d = 4 - \varepsilon$, and treated ε as a small continuous parameter. The perturbation technique has proved extremely powerful for predicting the values for critical exponents and their variations with the symmetry properties of various systems.

The fact that life, at least as far as critical phenomena are concerned, becomes much simpler in the fourth dimension is not instantly comprehensible to many people. Fisher explains it in the following way. If a person walks at random on a line or on a surface, he is certain to encounter any lamppost on the line or fall into any hole in the surface if he keeps walking long enough. But if a person walks around in three dimensions, for example, on a mammoth lattice or jungle gym, there is already a finite chance that he will miss some point. So there is a major difference between two and three dimensions. Now if two people are moving around in two or three dimensions, their paths are certain to cross, if they walk long enough. But in four dimensions, not only is there a good chance the walkers never will meet one another, but they have a finite chance of never seeing any trace of the other's path, no matter how long they walk. This indicates that in four dimensions processes that happen in different parts of the space—even if they spread through it—are more independent than in three or fewer dimensions.

The recent work of Wilson has drawn together explanations of many diverse physical effects into a coherent and esthetically pleasing theory. However, the work may have an even wider impact because the behavior of substances near critical points is analogous to many other problems. According to Kadanoff, studies of the growth of polymer chains, the Kondo problem (how a magnetic impurity behaves in a metal), well-developed turbulence, and the many particle interactions that are observed in high-energy physics may all yield way when approached with the same mathematical techniques that have unified the understanding of critical phenomena.

—William D. Metz