Operations Research: Some Contributions to Mathematics

Applied mathematics gets a new surge of life from techniques of operations research.

Thomas L. Saaty

Bertrand Russell once defined mathematics as the "subject in which we never know what we are talking about nor whether what we are saying is true" (*1*). In this definition he is not questioning the validity of mathematical thinking but its applicability to the real world. I have applied the same type of definition to operations research: "The subject in which we never know the real problem we should be talking about nor whether our solution of it has any relevance to reality." Nevertheless, we do such research because people have problems and, as scientists, we believe that any model is better than none; it is all right to give bad answers to problems if worse answers would otherwise be given.

Operations research is a field of science concerned with developing ideas and methods to improve decision-making. Decision-making involves the identification of values, objectives, priorities, means, resources, and constraints under conditions of certainty or uncertainty for short- or long-range local or global purposes. Since organizational structure is an important aspect of a decision process, it is also necessary to consider information flow, level and type of participation, and the socioeconomic framework. Decision rules are then introduced and methods (qualitative or quantitative) are developed to apply these rules. It is mostly these methods that bring decision-making close to mathematics by borrowing ideas from it and frequently expanding and enriching these ideas. In recent years we have become aware of a lack of basic mathematical concepts needed to make better decisions in complex social, political, economic, and related problems. Some people have been attempting to remedy this problem, but progress has been slow and difficult.

Individuals trained in the tradition of the modern school of pure mathematics have the tendency to assume that significant contributions to the development of their subject can only come from research conducted by pure mathematicians on structures gradually evolved and nurtured by them. Therefore, some may infer that very few fundamental contributions to mathematics can come out of applications; they are always ready to defend mathematics because they fear that its loftiness and beauty may be marred by admitting the suspicious and sometimes unorthodox practices of those who apply the subject to the real world. But mathematics has proven itself a tool for better understanding; the concepts of mathematics as a formal discipline and as a tool provide mutual growth and enrichment of the field.

If one divides the pursuit of the development of mathematical models into (i) finding bounds on solutions, (ii) existence and uniqueness, (iii) characterization, (iv) construction, (v) convergence, and (vi) approximation and error, then operations research contributions to mathematics have occurred mostly in the areas of characterization and construction. In addition, the wide use of athematical models in applications has helped draw attention to a number of areas of mathematics that had not previously received such close scrutiny or thorough treatment. Many problems involving stochastic processes fall into this category. Examples are search theory, inventory theory, and queueing theory.

A list of operations research contributions to mathematics, and in particular to applied mathematics, would include linear and nonlinear programming, stochastic programming, queueing theory, search theory, inventory theory, scheduling, decision and value theories, game theory, differential games, dynamic programming techniques, optimization in integers, network flows and network design, algorithms, and the new concept of fuzzy sets. However, these contributions may be divided into two basic categories: those concerned with stability and those concerned with optimization. One can unify the various methods of operations research within these two fundamental concepts underlying the use of mathematics in the real world (*2*).

Operations research contributions concerned with stability include those that define and characterize the conditions under which stability of equilibrium exists. Contributions concerned with optimization include those in which theories and algorithms, formal and heuristic, are developed for the purpose of selecting from the many stable policies one that is optimal. Although maxima and minima are particular examples of equilibria in an extended framework, I prefer to separate stability and optimality for emphasis. Even in the field of probability the measures used to study problems such as the mean and standard deviation are stability concepts. Stability is also the explicit focal concept of game theory. In other areas stability is very important but its role is disguised. In game theory, where there are several interests that may be in conflict, a resolution requires that they each be satisfied but within the general context, that is, that each interest must acknowledge that the other interests are entitled to a share of the claim.

Policies in general, stable or unstable, are studied in the field of single interest optimization by examining solutions of the constraining set, for example, a set of differential equations or inequalities representing a dynamic system. For optimal policies one must find those (one or more) policies that yield a maximum or a minimum of a given function. Because decision processes require implementation, the construction of stable optimum policies, for example, is a pressing task. Algorithms play a significant role in this pursuit of construct-

The author is professor in the graduate groups of operations research, applied mathematics, and peace science at the University of Pennsylvania, Philadelphia 19104. This article is adapted from a lecture presented at the AAAS annual meeting, 27 December 1971 in Philadelphia.

ting solutions. Also, it is not surprising that algorithms have flourished in the general field of optimization, particularly in the discrete setting of graphs and networks. The continuous setting in Euclidean space has been illustrated in one of its simplest and most elegant forms by Tucker (*3*).

In this article I discuss algorithms as they may be applied to the construction of stable policies; game theory as a means of analyzing the stability (or instability) of conflict, and for analyzing optimum policies; and optimization in a geometric setting. I use fuzzy sets to illustrate recent attempts to change some of the basic formal concepts in directions more suited to the types of applications now encountered particularly when the human element is included in the model. Thus I indicate a variety of problems to which operations research can be applied, rather than justify that all operations research concepts revolve around the fundamental idea of stability.

The Scope of Operations Research

The two main types of research conducted by operations research scientists are illustrated in Fig. 1. Tool-oriented research is concerned with the development of applicable basic mathematical tools and may be divided into two areas, continuum and discrete mathematics. Under continuum mathematics there is a triangle whose corners denote three fields always in use in operations research and whose sides indicate interactions among the three fields. The first of these fields is concerned with solutions of equations and with inequalities that commonly occur, particularly as constraints in the field of optimization. These equations and inequalities may be algebraic, differential, integral, difference, or hybrids of them, and are the basic tools in constructing descriptive models. Optimization, the field in which normative or prescriptive models are constructed by selecting optimum policies, is central to the use of operations research in decision-making (see Fig. 2). Optimization usually requires the use of an objective function that must be maximized or minimized subject to constraints.

In the third field, both descriptive and normative models may be deterministic or probabilistic, the difference depending mainly on whether or not the parameters of the problem are given according to probability distributions. Another way in which probability could enter a problem is to have an entire inequality constraint conditioned to hold with prescribed probability.

For simplicity, I have also divided discrete mathematics into three fields. The first field, combinatorial mathematics is concerned with arrangements of elements into sets and, particularly, with the existence of numbers and types of certain configurations. Closely related to combinatorial theory is graph theory, concerned with the study of relations between sets. Graph theory has been used in a number of applications of mathematics to the social and physical sciences. In network flow, a graph is used as an underlying structure for the analysis of such questions as the maximum flow of materials from sources to sinks in the graph.

Fig. 2 (facing page). Optimization, an outline. [From Saaty, 1970 (*2*)]

The second field of discrete mathematics is concerned with optimization over discrete sets involving diophantine equality and inequality constraints. The rapid growth of the field of integer programming with its wide variety of algorithms has contributed greatly to the expansion of this field. Geometric number theory is concerned with such problems as packing a maximum number of prescribed sets into a given set or covering a set with a minimum number of such sets with every point of the given set being covered at least once. As an example of the application of covering, radar stations may be located so that every point 50 miles (80 kilometers) up or less is under the surveillance of at

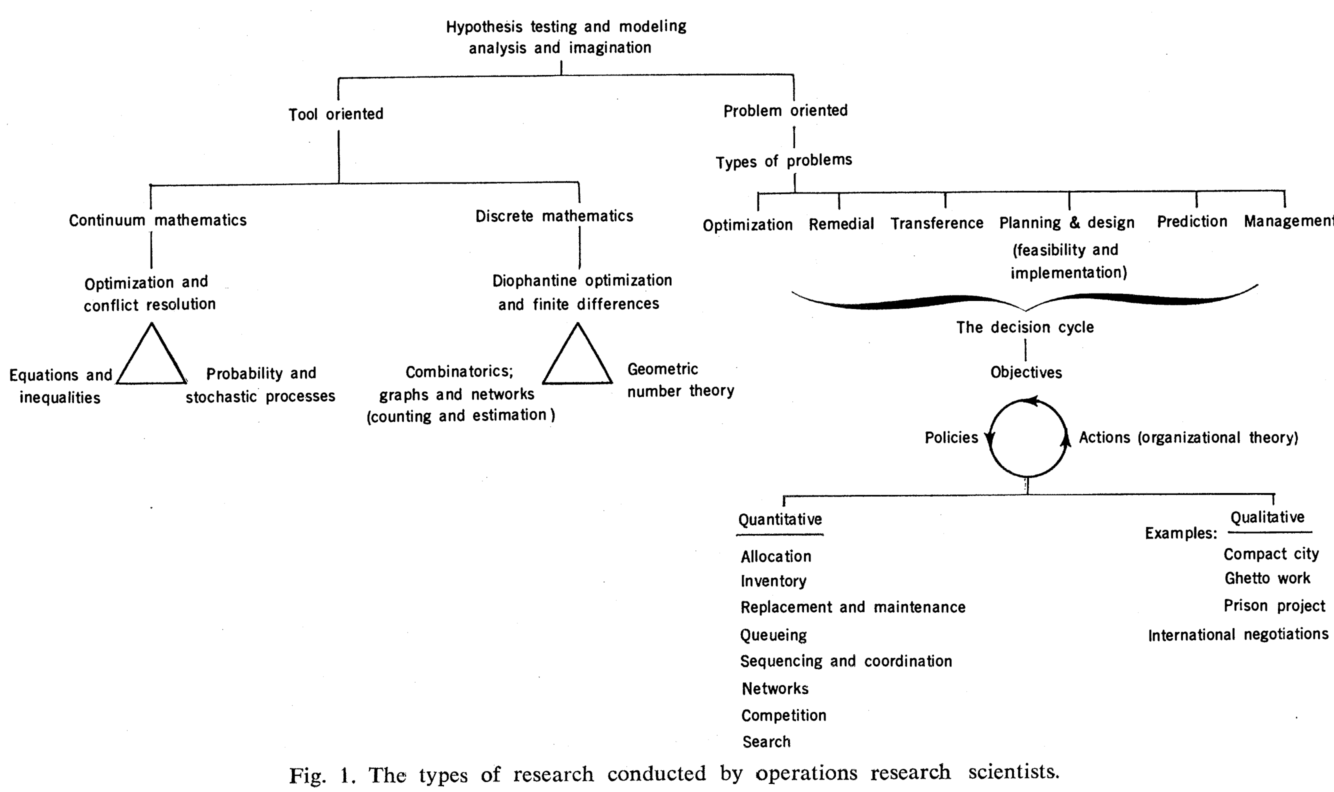
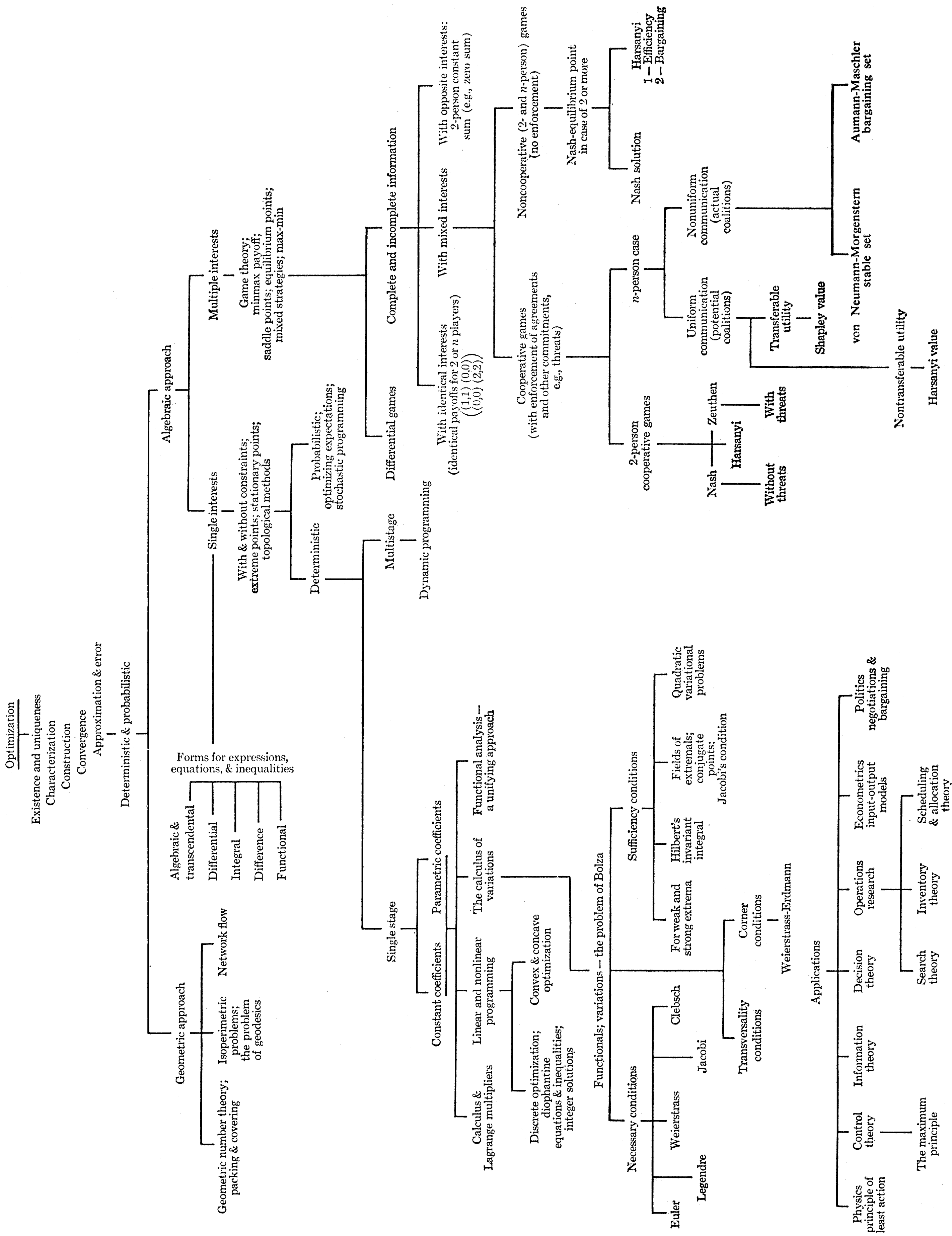


Fig. 1. The types of research conducted by operations research scientists.


Optimization
Existence and uniqueness
Characterization
Construction
Convergence
Approximation & error
Deterministic & probabilistic
Geometric approach
Geometric number theory; packing & covering
Isoperimetric problems; the problem of geodesics
Network flow
Forms for expressions, equations, & inequalities
Algebraic & transcendental
Differential
Integral
Difference
Functional
Algebraic approach
Single interests
With & without constraints; extreme points; stationary points; topological methods
Deterministic
Probabilistic; optimizing expectations; stochastic programming
Single stage
Constant coefficients
Parametric coefficients
Calculus & Lagrange multipliers
Linear and nonlinear programming
Discrete optimization; diophantine equations & inequalities; integer solutions
Convex & concave optimization
The calculus of variations
Functional analysis — a unifying approach
Functionals; variations — the problem of Bolza
Necessary conditions
Euler
Legendre
Weierstrass
Jacobi
Clebsch
Transversality conditions
Corner conditions
Weierstrass-Erdmann
Sufficiency conditions
For weak and strong extrema
Hilbert's invariant integral
Fields of extremals; conjugate points; Jacobi's condition
Quadratic variational problems
Multistage
Dynamic programming
Multiple interests
Game theory; minmax payoff; saddle points; equilibrium points; mixed strategies; max-min
Differential games
Complete and incomplete information
With identical interests (identical payoffs for 2 or n players) ((1,1) (0,0) / (0,0) (2,2))
With mixed interests
With opposite interests: 2-person constant sum (e.g., zero sum)
Cooperative games (with enforcement of agreements and other commitments, e.g., threats)
Noncooperative (2- and n-person) games (no enforcement)
Nash-equilibrium point in case of 2 or more
Nash solution
Harsanyi
1 — Efficiency
2 — Bargaining
2-person cooperative games
Nash
Harsanyi
Zeuthen
Without threats
With threats
n-person case
Uniform communication (potential coalitions)
Nonuniform communication (actual coalitions)
Transferable utility
Shapley value
Nontransferable utility
Harsanyi value
von Neumann-Morgenstern stable set
Aumann-Maschler bargaining set
Applications
Physics principle of least action
Control theory
The maximum principle
Information theory
Decision theory
Operations research
Search theory
Inventory theory
Scheduling & allocation theory
Econometrics input-output models
Politics negotiations & bargaining

least one radar set. Applications of this field to crystallography are relatively well known.

Operations research could not have become established as a scientific discipline without its practitioners insisting on quantification. The philosophy of total systems has broadened the boundaries of operations research to the extent that parallel mathematical disciplines must now be developed to support the new philosophy. Unless there is progress in mathematics, operations research will encounter difficulties not the least of which will be lack of rigor, a problem that is common in other sciences having social applications. By rigor I mean the finality and conclusiveness that is inherent in mathematical thinking and that guides the way from complexity to simplicity and from many to few alternatives. Operations research needs mathematics if it is to survive as a scientific discipline. Disciplined imagination is not enough. On the other hand, new mathematics must be invented to satisfy the scientific needs of operating research. "Theorem proving" would be even more valuable than "solving." Thus we need more characterizing than constructing.

The most exciting aspect of operations research to its practitioners is that they are constantly becoming acquainted with the details of many disciplines. To solve problems, a variety of which are shown in Fig. 1, and to obtain solutions suitable for decision-making, useful mathematical tools have been developed for the areas listed under quantitative. However, the solution of most problems requires a broad framework of logic, imagination, and quantification. Four examples of such qualitative problems are shown in Fig. 1.

The decision cycle emphasizes the fact that in addition to knowing his objectives and designing optimal policies to attain them, the operations research scientist is always faced with the very strong and relevant issue of implementation. A theoretical plan may be perfectly conceived, but there may be no way on earth by which it can be put into practice. This is reminiscent of the choral movement of Beethoven's Ninth Symphony. Presumably, the original version was very difficult if not impossible to sing but was close to the perfect scheme the composer had in mind. Because reality involves people, the composer had to compromise his structure for the final version which is still known for its vocal difficulty. Similar problems occur with idealized solutions. Very often it is more difficult to develop a plan for implementing a policy than it is to obtain a mathematical solution prescribing the optimal policy; even in operations research there are people whose esthetic standards do not permit a flexible transition from optimum policies to reality.

Various types of mathematical questions may be pursued, ranging from existence to construction (see Fig. 2). There are two general methods of approach, geometric and algebraic, and the use of each depends on the type of problem to be solved. Although many problems can be studied by the algebraic approach, there are others that cannot be completely divorced from their geometric setting.

The algebraic approach may be divided on the basis of problems having a single interest (that is, one objective function, or several objective functions combined in a unified utility framework) or having multiple interests involving conflict and cooperation. The subdivisions under single interest optimization indicate the various types of constraints and objective functions that are related to such problems and the mathematical approaches to analyzing them. Most of these subjects are now being studied from the standpoint of functional analysis in abstract spaces that provides them with considerable unity.

Generally speaking, the field of mathematical programming is concerned with algebraic expressions, the calculus of variations with integrals, control theory with differential or difference types of constraints, and stochastic optimization with any of the foregoing. All such problems may occur under various probability assumptions and the object is to obtain a point, or a function, as the case may be, which yields a maximum or a minimum to the objective function and also satisfies the constraints.

Multiple interest optimization is a young formal discipline studied primarily through the theory of games which I discuss later in this article.

Algorithms

An algorithm is a step-by-step procedure consisting of three main stages: (i) examination of all possible movements, (ii) selection of one movement according to a set of rules, and (iii) decision as to whether to stop or continue. The rigorous algorithm must be accompanied by a mathematical proof of its convergence to the optimal solution in a finite number of steps. In elementary terms, an algorithm is a procedure with well-defined rules followed systematically in order to obtain in a finite number of steps the solution to a given problem. An algorithm may converge to the true optimum solution in a reasonable number of steps, a highly desirable condition, but many problems require large numbers of iterations to achieve optimality. In such cases the cost of continuing the algorithm must be compared to the cost of truncating the process at a certain point, and thus the ability to estimate the error incurred by truncation is essential. There are also instances when algorithmic procedures are used to produce not the true solution but bounds on the solution.

In addition, there are problems that can be solved by heuristic algorithms, but such methods may not permit verification of all steps of the analyses. Such intuitive algorithms may not be ideal, but as initial approaches they can lead to iterative processes that are more rigorous. In many cases, heuristic algorithms are the only ones possible in difficult situations.

The importance of a rigorous algorithm lies in its adaptability to other forms; a clearly defined procedure can be translated into charts and then into programs. A Turing machine was invented in the 1930's. It is an elegant abstraction of the intuitive concepts of algorithms. This machine may be said to operate on an alphabet, or any other set of symbols, and on the possible movements of these symbols right and left, by defining the actions and movements associated with each possible state of the system. The states are represented by boxes in a flow chart of an algorithm, and the programs (instructions) for each state represent the movements of the symbols from box to box.

A search in a labyrinth provides one of the simplest examples of an algorithm. The typical setting for such a problem is that of Theseus and his girl friend, Ariadne (Fig. 3). Ariadne waits for Theseus while he wanders through a labyrinth in search of the Minotaur. If Theseus is allowed to walk twice over every path between two points, he can visit every junction of the labyrinth and remain entirely ignorant of the plan of the labyrinth. This duplication of paths is not necessarily the shortest way of reaching the Minotaur, but, if performed according to plan, the entire maze is traversed without Theseus losing his way.

With Ariadne at the entrance holding the start of a spool of thread, Theseus unwinds the thread as he traverses the labyrinth. To aid Theseus' memory (although this is not necessary) a corridor that is traveled once (the thread passes through it once) is lit with a yellow light, traveled twice (the thread has been rewound from it) is lit with a red light, and untraveled is lit with a green light. Thus, Theseus, starting at some position or vertex with Ariadne, unwinds the thread as he goes along green corridors which now turn yellow. If, in his meanderings he should revisit a junction, he must rewind the thread back to the immediately preceding junction, and the light in that corridor now changes from yellow to red. Note that Theseus may arrive at the Ariadne junction and not have to consider stopping, because the circuit and green corridor conditions must be checked first.

The algorithm is finite because no red corridor can be traversed; thus, the number of iterations cannot exceed twice the number of corridors. Unless he reaches the Minotaur, Theseus will stop at Ariadne with all corridors red. Every other junction is entered through a corridor, after which it turns yellow, and must be left before that corridor turns red. Ariadne's junction behaves differently in that when it is left it generates a yellow corridor and only when Theseus returns to it will it turn red. Thus Theseus will be stranded at a junction with all corridors red only when that junction is Ariadne's.

If we consider the junctions of the labyrinth to be vertices of a graph, and constrain them to be equidistant in the four directions, we can create a Turing machine program equivalent to the algorithm. By allowing movement in four directions and by using an appropriate alphabet to describe the conditions of the corridors we can program the solution. Both the Turing program and the conventional form of the algorithm follow the basic logic according to which the possible moves (corridors) are examined, one of them is chosen, and the subsequent situation is evaluated. This procedure can be generalized for certain types of algorithms, known as branch-and-bound, by defining three principles: an evaluation function, a separation principle, and a stopping rule. The first assigns a value to any step of the algorithm; this value usually depends on the availability of a solution at that point. The second separates the problem into the possible movements for consideration; this separation occurs successively throughout the process until the optimal solution is obtained. The third states the conditions for terminating the algorithm.

The method of this generalization remains valid for any iterative process. Not only does it relate a large number of algorithms by a single framework but also serves as a format which provides guidelines for creating new algorithms. Thus, we have established generalized frameworks among iterative problem-solving methods, rather than a disjoint set of problems and solutions.

Studies of the following problems were initiated by operations research scientists and have greatly helped the growth of algorithms. Such algorithms have directed applied mathematics away from calculus and differential equations.

The shortest route algorithm is an integral part of another algorithm for a network flow. In a network with certain capacities and costs, the cost of sending a commodity from an origin to a destination is minimized by sending the maximum flow along the least costly route. Similarly, the longest route algorithm solves the PERT (program evaluation and review techniques) scheduling problem, of Polaris submarine construction fame. By finding the longest times necessary to complete a sequence of tasks, manpower and resources can be shifted to those tasks that are identified as bottlenecks or are found to be otherwise restrictive (*4*).

The problem of the traveling salesman is widely known in operations research. The salesman wants to visit his clients by covering the shortest total distance without repetition. The problem can only be solved effectively for the salesman wanting to visit a small number of cities; for a larger number of cities the approach has so far been heuristic.

Stability and Game Theory

Perhaps one of the most maligned and least appreciated fields of operations research and mathematics is the theory of games. In fact, operations research scientists themselves have not fully appreciated the uses to which can be put interpretations of human multiperson conflicts by means of game theory.

Game theory was developed to fill a need for structuring and defining methods for solving problems in which several objectives, representing the goals

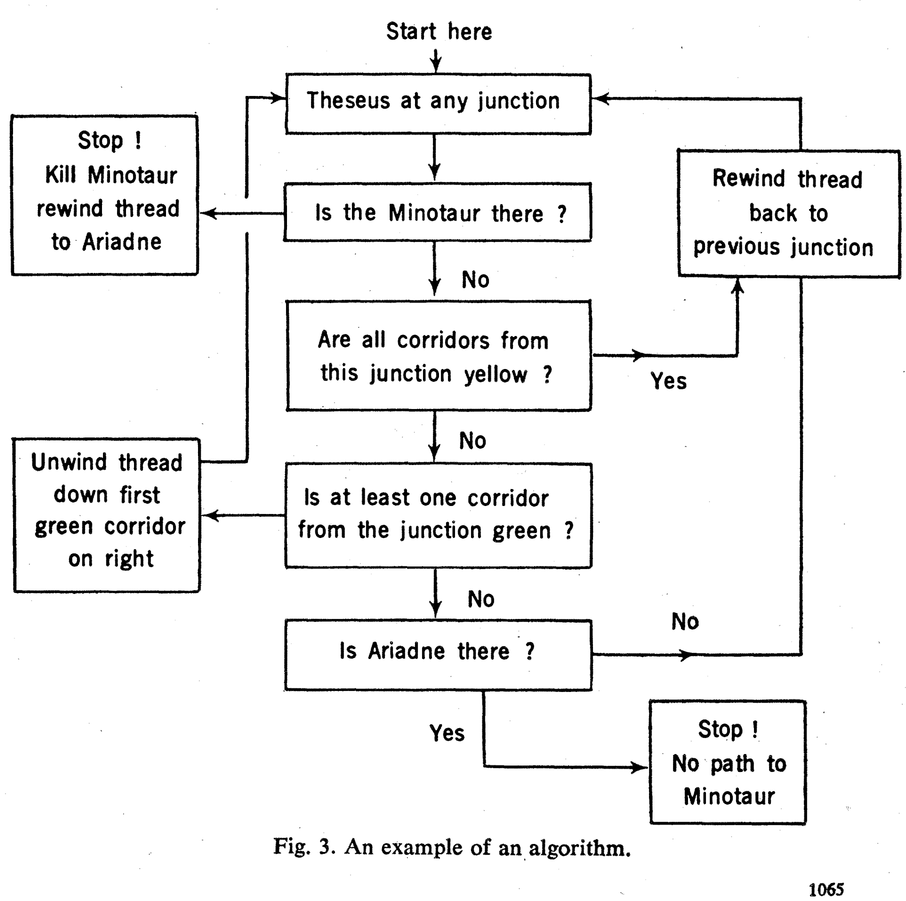


Fig. 3. An example of an algorithm.

of various people, must be optimized simultaneously. Each of the objectives may be subject to different sets of constraints if the parties involved do not wish to be completely cooperative in accepting the joint solution of a straight optimization approach. The logic of game theory, frequently intricate, permits a better understanding of the nature of the problems to be solved. Thus, while game theory seeks to be prescriptive or normative, its contributions to problem-solving result from the logical approach it employs—not from the numbers it uses. There are fair-division games, bargaining games, quota games, and many other kinds of games.

The theory leading to the concept of a solution for zero-sum two-person games, each person having a finite number of strategies, was developed by von Neumann (*5*). Such games are abstractions of most parlor games. They are noncooperative conflicts in which what is gained by the winner is completely taken from the loser. According to this theory, any such game has a pure or a mixed strategy solution. A game with perfect recall, such as chess or checkers, has a solution in pure strategies—that is, there is one best strategy always to be pursued, its result can be a win or a draw, and that is how these games always are. The players know all the preceding moves and all moves are a result of behavioral choice rather than a result of the intervention of chance. There are many concepts of solution for nonzero-sum games. In fact we still need a complete theory of zero-sum n-person games where n is greater than two. Coalitions play a central role in games involving more than two people.

For nonzero-sum n-person cooperative and noncooperative games with a finite number of strategies a general theory was started by Nash (*6*). The idea of stable equilibrium plays a central role in this theory. A stable solution for a noncooperative game is a payoff, and its corresponding strategies are such that, if all players but one adhere to their strategies and one of them attempts to play some other strategy, he cannot improve his own payoff. Although Nash's theory encountered some dilemmas, it continues to be one of the simplest and most readily acceptable definitions of what might constitute a solution for certain problems encountered in real life.

A number of other concepts of solution have been examined. Among these is the idea that an index may be used to measure the power that an individual can exert in obtaining a win in a coalition which he might join. In a bargaining situation the players, by a mixture of threats and cooperative proposals, attempt to convey to each other how they think stability would be achieved. Offers, counteroffers, and compromises usually lead to a resolution that is stable for a limited period of time.

Game Theory and Arms Control

A very simple illustration of the application of a game matrix to a bargaining situation in which the information possessed is incomplete is a game portraying the escalation of arms through MIRV's (multiple independently targeted reentry vehicles). As a possible arms control measure a few years ago, the United States hoped that the Soviet Union would not develop MIRV's. The payoff matrix for such a situation might have the form shown in Fig. 4. In each pair of numbers the first number indicates payoff to the United States and the second to the Soviet Union. The conclusion is that there are situations in which an opponent cannot be guided at all by what one party says because, regardless of what is said, it may be in the opponent's interest to do the opposite. For the opponent to do what has been requested it is more important to supply the opponent with greater factual information about what the situation really is than to continue an argument.

Figure 4 shows that to avoid a -4, the United States, if it has MIRV's, might try to convince the Soviet Union that it does not have them. However, it would be better for the Soviet Union to have them because then they would get -4 instead of -5. If, on the other hand, the United States has no MIRV's, it might try to convince the Soviet Union not to have them; otherwise, the United States would get -5 instead of 0. But, in that case, it would still be better for the Soviet Union to have them because then they would get 1 instead of 0. Thus, nothing that the United States tells the Soviet Union should influence the Soviet Union's judgment about MIRV's because, given this payoff matrix, it is better for them to have MIRV's no matter what is said. The decision must be based on other information; for example, some inspection scheme by both sides to ensure that neither side has them. Of course, we know from the newspapers that both sides know how to make MIRV's and may even be mass producing them.

A Coalition Game

As a second illustration consider the following coalition game. Seven people are divided into two groups, a group of five and a group of two. Some subset of the two groups must form a coalition and bargain the division of a given sum of money. If the coalition can stay together without being broken up by a member not in this coalition, then the game is terminated and the money is distributed among the members of the coalition according to how they bargained this division. Not all coalitions are allowed. An admissible coalition could be any one of the following nine. The group of two members with at least one and up to possibly all members of the group of five. This makes five coalitions. Then one member of the group of two with three, four, or five members of the other group can form a coalition. Finally the entire group of five can form one coalition. Thus in all there are nine coalitions. The problem is to devise a method of settlement that takes into consideration the bargaining power of each individual to form a coalition. That there is such an index of measurement was demonstrated by Shapley (*7*) who allocated each individual a part of the total sum of money depending on his "voting" power to form a winning coalition. In real-life bargaining, personality plays an important role which as yet cannot be included in the mathematics. In practice one encounters such bargaining games frequently and any familiarity with the mathematics makes it easier to understand them. By participating in such games the players learn to appreciate the difference between the power they have as part of the structure of the game and the power they have as behavioral members of a bargaining process.

Preferences and Sanctions

The following application of game theory has been used frequently in the field of arms control and disarmament and, more recently, in the analysis of the subway-highway debate in Washington, D.C. It has also been used in an attempt to make an analysis of the Middle East crisis. The approach is based on Nash's concept of solution (*6, 8*).

What each of the participants prefers among the several possible outcomes must be known. Often an outcome may be described by listing the actions (called options) available to each party and then stating whether that party takes or does not take the action for the outcome considered. If necessary, each option can be subdivided into more options. Thus the stability of the outcomes is tested on the basis of informed judgment on the preferences of each of the parties.

An outcome is potentially stable for a particular player (participant) if any action he might take to improve his position can be responded to by a sanction wielded by the other players—this being a joint action such that, whatever the particular player does, he will be in a position less preferable to him than the initial outcome. A potentially stable outcome will be actually stable for a player if he finds it credible (that is, if he does, in fact, believe) that a sanction would be applied if he moved away to a preferred position.

Finally, an outcome as a whole is stable if and only if it is stable for each individual player. Using this definition, the investigator proceeds by first listing the options available to each player as described above. Then he starts his analysis by selecting what appears to be a possibly stable outcome and examining it from the viewpoint of each player in turn to decide whether it is stable for this particular player. First he lists all possible unilateral changes this player can bring about and decides whether or not each would lead to a preferred outcome. If there exist any which would lead this player to a preferred outcome, the investigator examines all possible sanctions the others could use against such a move and decides whether each sanction would yield only "not-preferred" outcomes.

An outcome can also be examined from the viewpoint of coalitions of individual parties. A coalition (that is, a particular subset of parties) is defined as preferring one outcome to another if and only if each coalition member has such a preference. The definition is designed to enable us to examine how groups of players can jointly reach preferred outcomes by joint action.

The technique assigns options to each player, and the analysis based on these options may not lead to any outcome that is acceptable to all the parties. To overcome this difficulty in a way that conforms to real situations, the same problem, with a larger number of options, may be examined to see if concessions could be made to the opposing party in order to reach a compromise on the outcome desired by the others. This is a method of embedding a negotiation problem in a larger one. With regard to conflicts that arise in the field of planning, these additional concession-options may be of a kind which require action at a future date. Thus, it may be that action is needed for some kind of development in the not-too-distant future that was being sought by one of the parties, while this same party objected to an urgent measure the remaining parties wanted. Then these parties might attempt to get a concession from the defector by offering him one of the outcomes for the future in which his interests are better served, thereby enlarging the scope of the game.

		Soviet Union	
		No MIRV	MIRV
United States	No MIRV	0, 0	−5, +1
	MIRV	+1, −5	−4, −4

Fig. 4. Application of game theory to a bargaining situation (MIRV, multiple independently targeted reentry vehicles).

Attrition-Attack Differential Game Model

Suppose that two nations, the United States and Vietnam, are engaged in a protracted war and have supplies of vital weapons at time t amounting to x_1 and x_2, respectively. Each nation can at any time choose to allocate its resources between "attrition," that is, depleting its enemy's rate of weapons supply, and "attack," that is, entering them in the major conflict. The accumulation of any excess weapons in attacks will ultimately decide the victor. Thus, the basic decisions are between the long-range policy of attrition and the short-range policy of direct attack (*9*).

Suppose the United States at time t split its force x_1 into the attacking component $(1-\alpha)x_1$ and that of attrition αx_1 where $0 \leqq \alpha \leqq 1$. Also, suppose that Vietnam, unimpeded, has the capacity to obtain weapons (by manufacturing them or obtaining them from allied nations) at the rate m_2 and loses them at a rate depending on the number αx_1 its enemy devotes to that purpose. Then

$$\dot{x}_2 = m_2 - c_2(\alpha x_1) \qquad 0 \leqq \alpha \leqq 1$$

where c_2 is a measure of effectiveness of the U.S. weapons against Vietnam's defenses. (Here dots over a variable indicate that its derivative is taken with respect to time.) A similar equation holds for x_1:

$$\dot{x}_1 = m_1 - c_1(\beta x_2) \qquad 0 \leqq \beta \leqq 1$$

If the war is to last some definite time, T, then the payoff, V, in terms of accumulated marginal superiority will be:

$$V = \int_0^T [(1-\beta)x_2 - (1-\alpha)x_1]dt$$

Here, T acts as a state variable with $\dot{T} = -1$.

If the partial differentials of V with respect to x_1, x_2, and T are denoted by V_1, V_2, and V_T, respectively, the optimal strategy is determined by:

$$\min_{\alpha} \max_{\beta} [(m_1 - c_1 \beta x_2)V_1 + (m_2 - c_2 \alpha x_1)V_2 - V_T + (1-\beta)x_2 - (1-\alpha)x_1] = 0$$

If the further condition that x_1, x_2 $\geqq 0$ is imposed, we can rewrite the expression in the brackets as

$$S_1 x_1 \bar{\alpha} + S_2 x_2 \bar{\beta} + m_1 V_1 + m_2 V_2 - V_T + x_2 - x_1 = 0$$

where

$$S_1 = 1 - c_2 V_2, \; S_2 = -1 - c_1 V_1$$

and

$$\bar{\alpha} = \begin{cases} 0 \text{ if } S_1 > 0 \\ 1 \text{ if } S_1 < 0 \end{cases} \qquad \bar{\beta} = \begin{cases} 0 \text{ if } S_2 < 0 \\ 1 \text{ if } S_2 > 0 \end{cases}$$

If we consider time in reverse ($\tau =$ constant $- t$) and differentiate, we find

$$x_1' = -m_1 + c_1 \bar{\beta} x_2 \qquad V_1' = S_1 \bar{\alpha} - 1$$
$$x_2' = -m_2 + c_2 \bar{\alpha} x_1 \qquad V_2' = S_2 \bar{\beta} + 1$$
$$T' = 1$$

where x' represents $\partial x / \partial \tau$.

To obtain the optimal strategies we complete the initial conditions.

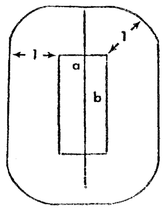
Along the terminal surface ϕ, $V = V_1 = V_2 = 0$ and, hence, $S_1 = 1$ and so $\bar{\alpha} = 0$; $S_2 = -1$ and so $\bar{\beta} = 0$. Thus the war concludes with both sides fully attacking.

Integrating the above equations for the optimal strategies using the initial conditions we get

$$x_1 = S_1 - m_1 \tau \qquad V_1 = -\tau$$
$$x_2 = S_2 - m_2 \tau \qquad V_2 = \tau$$
$$T = \tau$$

and, with these strategies,

$$S_1 = 1 - c_2 \tau \qquad S_2 = -1 + c_1 \tau$$



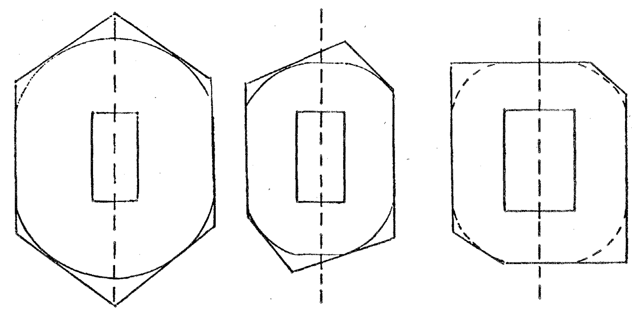


Fig. 5 (left). The "parallel domain" for a house base, used for determining the maximum number of houses that can be fitted into a given area. Fig. 6 (right). The three main types of hexagon that indicate the minimum areas that would be occupied by houses designated to satisfy minimum separation conditions.

which first become nonpositive when $\tau = 1/c_2$ and $1/c_1$. Since it is assumed that $c_1 > c_2$, $\tau = 1/c_1$ occurs first retrogressively. Thus at a time $1/c_1$ short of the end of the war, Vietnam switches from full attrition to full attack.

Further Examples of Operations Research Contributions

I will now give an example of how operations research models can lead to useful mathematical results.

The constraints of a linear programming problem define a convex set that is polyhedral in shape but may not be compact. The main idea is that the optimum of a linear program is on the boundary of this polyhedron and generally at a vertex. The simplex process, perhaps one of the most used algorithms for solving linear programming problems, enables one to move from one vertex to an adjacent one toward the optimum. The time it takes to solve the problem depends on the number of vertices. The number of vertices that a convex polyhedron can have cannot exceed the following binomial coefficient that is a sharp bound (that is, it is sometimes attainable):

$$V \leq \binom{F_{n-1} - \left[\frac{n+1}{2}\right]}{F_{n-1} - n} + \binom{F_{n-1} - \left[\frac{n+2}{2}\right]}{F_{n-1} - n}$$

where V is the number of vertices, n is the dimension of the space, and F_{n-1} is the number of hyperplanes of dimension $n-1$. (Brackets indicate the nearest integer.)

It is in the application context that operations research pushed forward the solution to this mathematically significant problem. It took about 20 years to obtain the result and to rigorously prove it.

Interesting applications of network analysis are discussed in the July 1970 issue of *Scientific American* (*10*). Optimum design, reliability, and the vulnerability of networks are all important concepts.

As an example of operations research applied to network analysis (*10*), I will describe the gas pipelines network from wells in the Mexican gulf to central storage stations at Pollock, Louisiana. A typical gas field may require a network that connects 25 wellheads; such a network may cost $100 million. The problem is to select a set of pipe diameters that minimize both the sum of investment and the operating costs that include the high cost of compressing the gas for delivery. The main trunk should be large enough to accommodate the additional flow that might result from the discovery of more wells, yet if it is built too large for today's needs and requires a large investment, it may turn out that the gas productivity from existing wells subsequently decreases and no new gas wells are found. There are seven prescribed pipe diameter sizes that may be used. With 25 branches and one main trunk there would be 7^{26} possible networks to choose from, a figure which defies computer capability. A "branch exchange" method has therefore been developed to select an economically good network. By this method a new branch is substituted for an old one and the flow throughout the network is recomputed to see if the cost has improved. The computations are repeated to explore new branches at all nodes of the tree. A number of improved network designs are in operation that would have cost millions of dollars more had they not been subjected to this analysis.

Geometric Approach

The following examples of optimization are based on the geometry of the problem to be solved and treated according to the theory of *The Geometry of Numbers* created by the mathematician, Hermann Minkowski, during the early part of this century (*11*).

There are many problems in real life that involve both geometric regularities and irregularities and it is sometimes difficult to decide on the best approach to their solution (*12*). To solve the problem of parking cars, for example, the same method may be used as that for solving the barrel-barrel problem, but allowance must be made for the turning angles of cars.

Let us consider the following questions. Given a large rectangular area, what is the maximum number of congruent unit circles or wine barrels (small relative to the total area) that can be packed in it so that any barrel can be moved without disturbing any of the others? How must they be arranged? We find that the barrels must be arranged in offset and touching double rows separated by corridors wide enough to move the barrels out. The density of this packing is

$$\frac{1}{2}\left(\sqrt{5}-1\right)\frac{\pi}{\sqrt{12}}$$

barrels per unit area.

A similar problem is encountered in the design of housing estates. As many rectangular houses as possible must be built on a large area and minimum separation conditions must be satisfied. We may take this minimum distance to be two units and scale the dimensions of the rectangular base to be a and b with $a < b$, $b = 1$.

We may create a "parallel domain" for the house base, as shown in Fig. 5, by drawing a boundary to contain all points of distance ≤ 1 from a point of the rectangle, inside the boundary. The problem then becomes one of finding the most efficient way to pack these sites into the given area (*13*).

For the initial calculations, a square base of side $2x$ is used for the house. Three main types of hexagon emerge as significant when a variety of possible hexagons are considered. When the areas of these hexagons have been determined the geometrical restrictions on x with repect to a make it possible to determine the arrangement of the hexagons that permit optimum use of the given area. The areas of these hexagons may be compared with respect to the value of x within the permissible ranges. Depending on which of the following relations holds: $2x \leqq 4 - \sqrt{12}$, $4 - \sqrt{12} < 2x < 2 - \sqrt{2}$, or $2x \geqq 2 - \sqrt{2}$, hexagons of types 1, 2, and 3, respectively, give the minimum areas (see Fig. 6).

The results may then be applied to rectangular bases, as it can be shown that a hexagon which is minimal for a square of side $2x$ is also minimal when the square is "stretched" to give a rectangle of shorter side $a = 2x$ (the hexagon is stretched along with the square). Thus hexagons of types 1, 2, and 3 will be minimal according as $a \leqq 4 - \sqrt{12}$, $4 - \sqrt{12} < a < 2 - \sqrt{2}$, and $a \geqq 2 - \sqrt{2}$, respectively.

Fuzzy Sets

In many fields of science, problems having an element of uncertainty and imprecision are conventionally treated according to the concepts and methods of probability theory. However, there are also situations in which the imprecision stems not from randomness but from the presence of a class or classes (that is, fuzzy sets) that do not possess sharply defined boundaries. Thus there are no precisely defined transitions that make these sets members or nonmembers of a class.

Such complex and ill-defined systems pervade the life and social sciences and are also found in the physical sciences where systems may be too complex to admit of analysis by conventional mathematical means. In general, the complexity of the system may increase to the point where it becomes impractical or infeasible to make precise statements about it. At that point, any meaningful assertion about the system must necessarily be fuzzy in nature.

The classes encountered in these situations are fuzzy in the sense that they do not possess sharply defined boundaries. In the case of a class with a fuzzy boundary, an object may have a grade of membership in it which lies somewhere between full membership and nonmembership. Zadeh called such a class a fuzzy set (*14*). Let $X = \{x\}$ denote a set of points. A fuzzy set A in X is a set of ordered pairs:

$$A = \{[x,\ \mu_A(x)]\}, x \in X$$

where $\mu_A(x)$ is termed the grade of membership of x in A. If $\mu_A(x)$ takes on values in a space M—the membership space—then A is essentially a function from X to M. The function $\mu_A : X \rightarrow M$ which defines A is called the

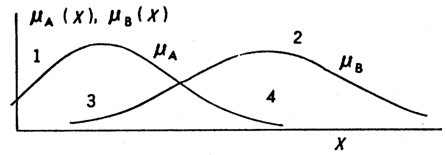


Fig. 7. Intersection and union of fuzzy sets.

membership function of A. When M contains only two points, 0 and 1, A becomes an ordinary set.

Some elementary properties of fuzzy sets which are obvious extensions of the corresponding properties of ordinary sets may be defined as follows:

Containment. Let A and B be two fuzzy sets in X. Then A is contained in B (or, equivalently, A is a subset of B, or A is smaller than or equal to B) if and only if, for all x in X, $\mu_A(x) \leqq \mu_B(x)$.

Equality. Two fuzzy sets are equal, $A = B$, if and only if $\mu_A = \mu_B$.

Complementation. A fuzzy set A' is the complement of a fuzzy set A if and only if $\mu_{A'} = 1 - \mu_A$.

Union. The union of A and B, denoted by $A \cup B$, is the smallest fuzzy set containing both A and B. The membership function of $A \cup B$ is

$$\mu_{A \cup B} = \max(\mu_A,\ \mu_B)$$

Intersection. The intersection of A and B is denoted by $A \cap B$ and is defined as the largest fuzzy set contained in both A and B. The membership function of $A \cap B$ is given by

$$\mu_{A \cap B} = \min(\mu_A,\ \mu_B)$$

The intersection and union of fuzzy sets in R^1 are illustrated in Fig. 7. The membership function of the union is comprised of curve segments 1 and 2; that of the intersection is comprised of segments 3 and 4.

Algebraic product. The algebraic product of A and B is denoted by AB and is defined by

$$\mu_{AB} = \mu_A \mu_B$$

Algebraic sum. The algebraic sum of A and B is denoted by $A \oplus B$ and is defined by

$$\mu_{A \oplus B} = \mu_A + \mu_B - \mu_A \mu_B$$

Relation. A fuzzy relation, R, in the product space $X \times Y = \{(x,y)\}$, $x \epsilon X$, $y \epsilon Y$, is a fuzzy set in $X \times Y$ characterized by a membership function μ_R which associates with each ordered pair (x,y) a grade of membership $\mu_R(x,y)$ ϵ R. More generally, an n − ary fuzzy relation in a product space $X = X^1 \times X^2 \times \ldots \times X^n$ is a fuzzy set in X characterized by an n − ary membership function

$$\mu_R(x_1, \ldots, x_n),\ x_i \in X^i,\ i = 1, \ldots, n$$

Composition of relations. If R_1 and R_2 are two fuzzy relations in X^2, then by the composition of R_1 and R_2 we mean a fuzzy relation in X^2, denoted by $R_1 \cdot R_2$ and defined by

$$\mu_{R_1 \cdot R_2}(x,y) = \sup_{v} \min\,[R_1(x,v),\ R_2(v,y)]$$

where the supremum is taken over all v in X.

Fuzzy sets induced by mapping. Let $f = X \rightarrow Y$ be a mapping from X to Y, with the image of X under f denoted by $y = f(x)$. Let A be a fuzzy set in X. Then, the mapping f induces a fuzzy set B in y whose membership function is given by

$$\mu_B(y) = \sup_{x \in f^{-1}(y)} \mu_A(x)$$

where $f^{-1}(y)$ denotes the set of points in X which are mapped by f into y.

Shadow of a fuzzy set. Let A be a fuzzy set in $X \times Y$, and let f denote the mapping which takes (x,y) into x. The fuzzy set in X which is induced by this mapping is called the shadow (projection) of A on X and is denoted by $S_x(A)$. The membership function of $S_x(A)$ is given by

$$\mu_{S_x(A)}(x) = \sup_{y} \mu_A(x,y)$$

where $\mu_A(x,y)$ is the membership function of A.

Conditional fuzzy sets. A fuzzy set $B(x)$ in Y is said to be conditioned on X if its membership function depends on x as a parameter. Denote the membership function of $B(x)$ as $\mu_B(y/x)$. If the parameter X ranges over a space X, the function $\mu_B(y/x)$ defines a mapping from X to the space of fuzzy sets defined on Y. Then, a fuzzy set A in X induces a fuzzy set B in Y which is defined by

$$\mu_B(y) = \sup_{x \epsilon X} \min\,[\mu_A(x),\ \mu_B(y/x)]$$

The concept of a conditional fuzzy set is similar to the concept of a conditional probability distribution. Note that the definition of $\mu_B(y)$ is similar to the conditional probability identity

$$P_B(y) = \int_x p_B(y/x)\, p_A(x)\, dx$$

In many instances, the operations of summation and integration involving probabilities correspond to the operations of taking the supremum (or maximum) of membership functions; multiplication of probabilities corresponds to taking the infimum (or minimum) of membership functions.

The theory of fuzzy sets is still in a state of theoretical development. Because of the many real-life situations to which fuzzy set theory seems to be relevant, there has been worldwide interest in it and also some criticism of it.

Conclusions

The discussion of fuzzy sets indicates that set theory should be extended to make it more suitable for the development of algebraic structures with wider applications. Stochastic optimization, a synthesis of the three areas of continuum mathematics, is a rapidly growing field particularly in the context of control theory. Applications of discrete mathematics have found a rich outlet in graph theory, particularly in the social sciences. However, there remains a wide gap between the sophisticated activities that are required for the development of optimum policies and the more difficult and intractable problems of organizational structures and performance.

The state of mathematics at present does not permit the development of general models by which the physical and economic aspects of a problem can be related to their social implications. So far, the total systems approach requires that the problem be partitioned with each partition being modeled separately; imagination and logic must then be used to combine the solutions so that they apply to the original problem. The social sciences are making rapid progress along various lines of quantification but significant mathematical theories that can be applied in this field have yet to be found.

Because of its undefined boundaries, operations research provides unlimited opportunities for pioneering work in mathematics. Operations research extends the roots of mathematics into the real world.

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NEWS AND COMMENT

AMA: Specialty Journals Must Lure Paying Subscribers

The American Medical Association (AMA) is trying to save money. A drop in dues-paying members (dues are $110 a year) and rising costs have forced the AMA, which everyone presumed to have unlimited wealth, to take stock of its resources. In the last 2 years, the organization has gone in the red to the tune of more than $3.8 million. This sizable deficit for a presumably flush outfit has lead many observers to the conclusion that the AMA is going broke. That isn't really true. But the AMA has set about trimming what it sees as the fat out of its operation in an attempt to make the future fiscally black.

Apparently the AMA realized as many as 4 years ago that a bit of fiscal restraint might be in order, inasmuch as its House of Delegates adopted a resolution calling for cost-cutting in June 1968. The first clear evidence that that resolution was taking effect came late last month, when the AMA met in Cincinnati.

The house, which is the governing body of the association, approved two actions proposed by the board of trustees to pare expenses. The first will put an end to the long-standing practice of allowing AMA members a free subscription to one specialty journal, in addition to the *Journal of the American Medical Association* (JAMA), and raises questions about the future of those journals. The second calls for the elimination of some of the AMA's councils and standing committees, whose cost apparently was not deemed justified by their productivity. Together, these actions will save an estimated $840,000.

Ever since 1909, when it began publishing the *Archives of Internal Medicine*, the AMA has been in the business of publishing specialty journals. Today it puts out ten (Table 1), with a combined circulation of about 180,000. Designed from the beginning to be academic journals, they have been untouched by the political turmoils that beset the organization, and they seem proud of that independence. Qualitatively, they vary but generally seem to be considered acceptable. As one specialty journal editor phrased it, "Actually, they are all very good journals, though each is not necessarily the top journal in its field." Three of them make money: *American Journal*