Reports

Lunar Ephemeris: Delaunay's Theory Revisited

Abstract. *Delaunay's reduced Hamiltonian of the main problem in lunar theory is checked against a new analytical theory based on Lie transforms. It is found to be correct up to order 9 with the exception of one error in addition at order 7.*

Meaningful analysis of dynamical data returned from lunar experiments (radar echoes, orbiters, Apollo missions, and laser retroranging reflectors) requires highly accurate lunar ephemerides beyond the standards achieved by traditional celestial mechanics (*1*). Experts are not unanimous as to the best approach toward that goal: a straightforward numerical integration (*2*), an improvement of Hill-Brown's semianalytical theory (*3*), a modified version of Hansen's special perturbations (*4*), or a completely analytical theory (*5*). The most efficient answer will likely be a coexistence of methods that compensate one another in their respective weaknesses.

Until very recently analytical solutions were at a disadvantage: they require an extraordinary amount of extensive and complex algebraic manipulations. One hundred years have now passed, and Delaunay's *Théorie du Mouvement de la Lune* (*6*) is still the only analytical solution pursued systematically beyond the first few trivial orders.

Soon after the computers proved successful in the scientific field, astronomers considered programming them to automate literal computations (*7*). In that perspective, reproducing Delaunay's work (*6*) appeared as the ultimate test of what software packages should be able to do for celestial mechanics (*8*).

Table 1. Corrections to Delaunay's formulas for the mean motions of lunar theory.

Mean motions	Delaunay's coefficients	Correct coefficients
$\dot{l}_0$	$\frac{629}{8} m^3\gamma^2e'^2$	$\frac{297}{4} m^3\gamma^2e'^2$
$\dot{g}_0$	$\frac{401}{16} m^3e'^2$	$\frac{99}{4} m^3e'^2$
	$-\frac{1297}{16} m^3\gamma^2e'^2$	$-\frac{1287}{16} m^3\gamma^2e'^2$
	$-\frac{1079}{64} m^3e^2e'^2$	$-\frac{1089}{64} m^3e^2e'^2$
	$\frac{883245}{512} m^5e'^2$	$1725\ m^5e'^2$
$\dot{h}_0$	$\frac{23}{32} m^3e'^2$	$\frac{33}{32} m^3e'^2$
	$-\frac{349}{16} m^3e'^2e^2$	$-\frac{693}{32} m^3e'^2e^2$

The main problem of lunar theory focuses on the system earth-sun-moon, with those bodies taken as mass points. It assumes that the sun moves on a fixed Keplerian ellipse around the center of mass of the pair earth-moon and that the moon, although it remains closely in the neighborhood of the earth, revolves around it on what is thought of basically as a Keplerian ellipse but perturbed by the solar attraction. The solar effect is developed in trigonometric series in four arguments: l' (the mean anomaly of the sun), l (the mean anomaly of the moon), F (the elongation of the moon from its ascending node), and D (the mean elongation of the moon from the sun). The coefficients in the trigonometric series are power series in five variables: m (the ratio of the mean motions of sun and moon), α (the ratio of the semimajor axes for the Keplerian ellipses of moon and sun), e (the eccentricity of the moon's orbit), γ (the sine of half the inclination of the moon's orbit), and e' (the eccentricity of the sun's orbit). Other quantities, like the mass ratios sun/(earth + moon + sun) and moon/(earth + moon) and the gravitational coefficient, are also retained as symbols, so that the results of an analytical development can be used to improve their numerical values from fittings to observations.

The essence of the solution consists in eliminating all four angles from the differential equations by an averaging technique. We have selected a perturbation algorithm based on Lie transforms (*9*).

We aimed provisionally at a theory complete to order 8, roughly one order beyond that of Delaunay. The averaging proceeded in four steps. The elimination of the monthly terms (that is, terms containing l), which is in principle a straightforward operation, was so vast a project that it had to be staged in six operations—each one a continuous run of 3 hours on an IBM 360-44 with a core of 32,000 words and two disks of 250,000 words. The subsequent elimination of the annual terms (that is, terms containing l' but not l) entailed a decrease in order of one unit through a division by m, elimination of the long-period terms (that is, terms containing neither l nor l') implied a division by m^2, thus a decrease in order by two units. The elimination of the annual terms produced a very long-period term $4l' - 2l - 2F + 4D$, whose existence escaped Delaunay's attention (*10*). All computations were carried automatically by means of a package of subroutines assembled to process echeloned series (*11*).

After obtaining the averaged mean motions to order 10, we compared our results with Delaunay's formulas for the mean anomaly, the argument of perigee, and the longitude of the node. But the constants of motion in our theory differ from Delaunay's constants; appropriate adjustments were made analytically by machine. The conclusions are truly startling. Delaunay worked at his theory without any assistance, by hand, for some 20 years continuously; his literal calculations cover two volumes in quarto of 400 pages each; he alone proofread them. Yet we recovered all terms to order 9 in Delaunay's Hamiltonian (*6*, chap. 6, p. 234) with the exception of the term in $m^3\gamma^2e'^2$. Its coefficient should be 33/16, but Delaunay gave it as being 23/16. The mistake is easily explained. In Delaunay's theory this erroneous term is the sum of three contributions:

$$-(45/16)m^3\gamma^2e'^2$$

brought by operation 52 (*6*, chap. 4, p. 122);

$$(147/32)m^3\gamma^2e'^2$$

brought by operation 53 (*6*, chap. 4, p. 123); and

$$(9/32)m^3\gamma^2e'^2$$

brought by operation 54 (*6*, chap. 4, p. 123). Later on (*6*, chap. 6, p. 234) they are added together, and at this moment Delaunay made his slip.

The error is not a misprint. It propagates consistently in the mean motions $\dot{l}_0$, $\dot{g}_0$, $\dot{h}_0$ (6, chap. 6, p. 237), as we show in Table 1. To Delaunay's Hamiltonian should be added the function $A = 5/8(\mu/a)m^3\gamma^2e'^2$; hence, we should recover the coefficients in the third column of Table 1 (as produced independently by our theory) by adding to Delaunay's coefficients the partial derivatives of A with respect to L, G, and H, respectively. We checked and did indeed obtain this result.

A number of authors using Delaunay's theory as a check for their theories (*12*) have reported possible errors in his work, but they never pinpointed them. Our conclusion is that, except for the mistakes we uncover here, Delaunay's Hamiltonian and mean motion $\dot{l}_0$ up to order 9 are faultless and the same is true for his mean motions $\dot{g}_0$ and $\dot{h}_0$ up to order 7.

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Antarctic Ice Sheet: Stable Isotope Analyses of Byrd Station Cores and Interhemispheric Climatic Implications

Abstract. *Oxygen- and hydrogen-isotope analyses from the core hole through the Antarctic Ice Sheet at Byrd Station define temperature variations over more than 75,000 years. Synchronism between major climatic changes in Antarctica and the Northern Hemisphere is strongly indicated. The Wisconsin cold interval extended from 75,000 to 11,000 years ago. Three intra-Wisconsin warmer phases were all colder than pre- or post-Wisconsin times, which suggests that North American and Eurasian continental ice sheets did not disappear at any time during the Wisconsin.*

Oxygen- and hydrogen-isotope analyses of ice samples from depths between 99 and 2162 m within the 2164-m core hole (*1*) of 1968 through the Antarctic Ice Sheet at Byrd Station (80°01′S, 119°31′W, 1530-m elevation) are shown in Fig. 1. The values plotted represent the deviation (δ) of oxygen ($^{18}O/^{16}O$) and hydrogen (D/H) ratios from the corresponding ratios for standard mean ocean water (SMOW). The δ values for both oxygen and hydrogen are here expressed in per mil (*2*, p. 214), even though the δ for hydrogen is not uncommonly given in percent because of its larger magnitude.

Each point on Fig. 1 represents the oxygen or hydrogen δ value of a homogenized strip sample ranging from 30 to 151 cm long taken from the core at intervals ranging between 33 and 62 m, except near the bottom where spot samples were taken. Two or more data points at a single level represent two or more adjoining strip samples. The $\delta^{18}O$ variations, up to 1.5 per mil, in adjacent samples are an expectable product of secular variation, as each homogenized sample represents several years of accumulated snow.

A plot of $\delta^{18}O$ against δD values in Byrd core samples fits a curve, $\delta D = 7.9\ \delta^{18}O$, with slightly different slope than curves obtained from other areas. Since the δD to $\delta^{18}O$ relationship is primarily dependent on temperature, such curves may ultimately prove useful in defining subtle differences and variations in ancient environmental conditions.

In the case at hand, the strong similarity of the oxygen and hydrogen curves (Fig. 1) testifies primarily to the reliability of sampling procedures, sample handling, and analytical methods. Since the two curves are consistent, interpretative comments are made solely in terms of $\delta^{18}O$ data.

Age of ice at various levels is estimated from measured accumulation rates and calculations of thinning through flow, as prescribed by Bader and Nye (*3*, *4*). A constant accumulation of 12 g cm^{-2} yr^{-1} of water is used for the Byrd Station accumulation area. This figure is based on 8 years of measurements made by one of the authors (A.J.G.) at a large number of snow stakes in the vicinity of Byrd Station. These data compare favorably with values of accumulation obtained by other observers who used different measuring techniques (*5*). The calculation of ice ages carries some qualifications. It has been necessary to assume an ice thickness that is constant and a vertical strain rate that remains unchanged. Further, no allowance can be made for possible changes in accumulation rates with past climatological variations. Ages near the bottom are the least reliable because of extreme thinning by flow. The calculations were made independently by A.J.G., largely before he had knowledge of the isotope variations. Our confidence in the results is buoyed by the age (11,000 years) calculated for samples at a depth of 1050 m.

The significance of the isotope curves is that they reflect the relative temperatures at which the water substance composing the samples was condensed. A lower—that is, a more negative—$\delta^{18}O$ or δD value represents a lower temperature. Readers should note that the plot (Fig. 1) is linear for depth but not for age; age increases at an accelerating rate with depth. The data below a depth of 1000 m are plotted against time in Fig. 2.

Only the uppermost part of the ice underlying Byrd Station originates from snow accumulating in the immediate vicinity. With increasing depth, the ice comes from increasingly remote and higher sources. Present terrain configuration and location of the ice divide suggest that none of the ice under Byrd Station is likely to have accumulated in environments more than 300 m higher or 2° to 3°C colder than Byrd Station. The gradual decrease in $\delta^{18}O$ values between depths of 100 and 1050