TECHNICAL PAPERS

An Inverse Decibel Log Frequency Method for Determination of the Transfer Functions of Psychobiological Systems[1]

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In previous papers, the writer has proposed the concept of transfer function for representing psychological and biological systems, in a manner analogous to its use for representing linear physical systems, such as servosystems and electrical networks (*9, 12*). It was pointed out that such transfer functions could, in principle, be derived on a purely empirical basis, in view of the fact that a one-to-one correspondence exists between the transfer function of a linear system and its response to specific input functions. The present paper describes a method by means of which the transfer functions of lumped constant linear systems can be derived from experimentally determined frequency response curves. Although developed primarily for use in deriving the transfer functions of the response systems of living organisms, it is applicable to any system that can be regarded to a first approximation as a lumped constant linear system, without regard to the type of components of which it is made up, whether living or nonliving.[2]

The method to be described rests on concepts and procedures involved in the so-called decibel (db) log frequency method developed for analysis and synthesis of servosystems and analogous networks. This latter method is hereafter referred to as the direct db-log frequency method in contrast to the method to be described in the present paper, which we designate as an inverse db-log frequency method.

As the names imply, the goals and procedures in the two methods are opposite to each other. In the direct method, an analytical transfer function, like that shown in equation (1), is given, and the problem is to plot the gain and phase frequency response curves corresponding to this function. In the inverse method a gain-frequency response curve (experimentally determined) is given, and the problem is to derive the analytic transfer function represented by this curve.[3] This latter procedure is equivalent to finding the differential equation representing a system from the specification of the solutions of the equation for a particular set of input functions.

$$P(s) = \frac{30s + 10}{2s^3 + 4.5s^2 + s} \tag{1}$$

As a preliminary to the description of the inverse method, a number of the terms and procedures involved in the direct db-log frequency method should be clarified—namely, transfer function, asymptotic frequency response curve, exact frequency response curve, and the db corrections which must be made for points in the asymptotic curve to derive the exact curve. For a fuller description of the direct method and discussion of these concepts, reference should be made to recently published expositions of servotheory (*2, 5, 6, 7, 8*).

A key concept is that of transfer function. It is a mathematical function representing the properties of a given system. The standard way of deriving it is to write the differential equation of the system, convert it to operational form through the application of the Laplace transformation or an equivalent procedure, and then find, by elementary algebraic operations, the ratio of the output to the input functions (*8*). Examples, in factored form, and with nondimensional parameters, are given in equations (2) and (3). Equation (3) is derived from Equation (2) by substituting $j\omega$ for the complex variable s. The transfer function, as given on the right-hand side of each of these equations, contains only the independent variable s (or $j\omega$), and time constants such as T_1, T_2, $\ldots T_k, \ldots T_n$, representative of the components of the system and their organization. The transfer function thus provides a compact representation of the system itself, and a means for predicting the response of the system to any arbitrary input disturbance or stimulus (*4, 8*).

$$\frac{Y(s)}{X(s)} = K \frac{(T_2 s + 1)}{s(T_1 s + 1)\,(T_3 s + 1)} \tag{2}$$

$$\frac{Y(j\omega)}{X(j\omega)} = K \frac{(j\omega T_2 + 1)}{(j\omega)\,(j\omega T_1 + 1)\,(j\omega T_3 + 1)} \tag{3}$$

In plotting frequency response curves by means of the direct db-log frequency method, three steps are involved: formulation of the transfer function in factored form (equations 2 and 3), construction of an asymptotic db-log frequency curve made up of straight line segments, and construction of an exact db-log frequency curve on the basis of corrections made in the asymptotic curve.[4]

In the factored form of the frequency transfer function

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[2] A lumped constant linear system may be defined as one that can be described by a linear differential equation with constant coefficients.

[3] The method described here relates only to the gain-frefrequency curve and not to the phase curve. In a minimum phase system, the phase curve can be derived from the gain curve (*1*). A number of alternative terms have been used in place of gain, such as amplitude and modulus. Thus the db-log frequency method has also been called the log modulus method (*2*). A gain curve, measured in db units and with a change in sign, may also be referred to as a loss or attenuation curve.

[4] The term *frequency*, as used in this paper, is taken to mean the angular frequency ω, measured in radians per sec. It is equal to $2\pi f$, where f is measured in cycles per sec.

(equation 3), factors of the following three types may occur: $(j\omega)$, $(j\omega T_k+1)$, and $(j^2\omega^2T_q^2+j\omega 2\zeta T_q+1)$, the last being a quadratic factor which equals the product of two single order factors with conjugate complex roots. Each of the various types of factor may appear in numerator or denominator and with integral exponents equal to or greater than one. There can be only one of the $(j\omega)$ terms present, but any number of the other two types of factor. The single order time-constant factors differ from each other only in the value of the time constant T_k.

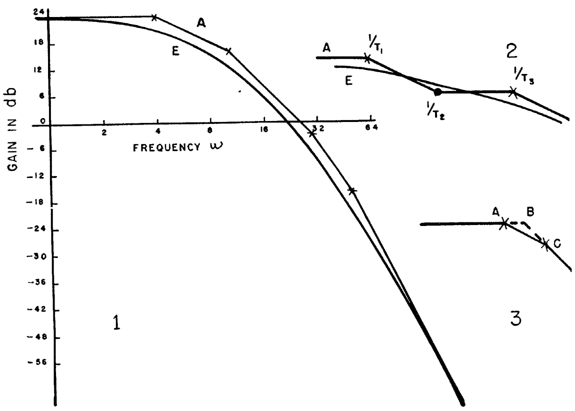
The straight line segments of which the asymptotic curve is composed all exhibit slopes of some multiple of 6 db per octave. The points at which they intersect are referred to as corner points or break points. There is one such corner point for each time-constant factor in the transfer function, the frequency coordinate of the corner point being equal to the inverse of the corresponding time-constant, e.g., $1/T_1$, $1/T_2$, and so on.

In one procedure for constructing the asymptotic frequency curve from the transfer function, one starts at the low frequency end with a line coincident with the 0 db axis, and continues it until the first corner point, located at ω equal to $1/T_1$, corresponding to the largest time constant. A second straight line segment is drawn from here to the next corner point, located at ω equal to $1/T_2$, T_2 being the next largest time constant, and so on. At each successive break, the following segment is drawn with a slope differing from the previous segment by $6n$ db per octave, where n is the exponent of the time-constant factor represented. The sign of the increment in slope is negative for factors in the denominator and positive for factors in the numerator. If there is a $(j\omega)$ term in the transfer function, the initial straight line segment has a slope of plus or minus $6m$ db per octave instead of 0 db per octave, where m equals the order or exponent of the $(j\omega)$ factor; and the slope is negative if the factor is in the denominator. This asymptotic plot is of great importance in connection with our present problem, since it places in evidence, at the various corner points, the values of the time constants which characterize a given transfer function.

An additional set of relations that we shall need to utilize in our inverse method relates to the db corrections (ε) that must be made at each corner point in the asymptotic curve in order to find the corresponding point on the exact frequency curve. The total correction at any value of ω equals the algebraic sum of the corrections due to *each* time-constant factor in the transfer function. The magnitude of the correction *for each factor* is given by equation (4). This quantity should be added to the asymptotic curve at any value of ω to obtain the corresponding point on the exact curve. Its sign is negative for factors in the denominator and positive for factors in the numerator. The smaller the interval between ω and $1/T_k$ on a logarithmic scale, the greater is the correction required for the factor containing T_k. Hence, if a set of corner points are densely distributed on the log ω axis, the net correction at the various corner points will be correspondingly high.

$$
\begin{aligned}
|\varepsilon| &= 10\log_{10}(\omega^2T_k^2+1),\ \text{for}\ \omega T_k \leqq 1 \\
&= 10\log_{10}(\omega^2T_k^2+1) - 20\log_{10}\omega T_k,\ \text{for}\ \omega T_k \geqq 1
\end{aligned} \tag{4}
$$

We may now consider the inverse method. Its design and the justification for the various steps proposed rest on the following set of principles. The over-all *asymptotic* curve of any transfer function is the algebraic sum of the asymptotic curves representing each of the component factors in the transfer function, due to the fact that these curves are logarithmic plots (in db) of the factors in the transfer function.[5] Similarly, the over-all



FIGS. 1, 2, and 3. 1—Relation of asymptotic (A) and experimental (E) frequency curves corresponding to a transfer function with only real poles. 2—Relation of asymptotic and experimental frequency curves corresponding to a zero lying between two poles in frequency. (Location of poles indicated by ×, zeros by ●.) 3—Procedure for finding approximate location of a second order corner point (B) from two single order corner points close together (A and C).

exact db-log frequency curve is the sum of the exact curves of each of the components. Hence the total or net correction representing the difference between the over-all exact curve and the over-all asymptotic curve will equal the sum of the correction curves for each of the components.[6] Our aim has been, therefore, to find a way of determining the total correction curve (or appropriate points on it), and to subtract it from the experimentally obtained exact frequency curve. This procedure, if accurate, will lead to the asymptotic curve and thus reveal, by way of the corner points, the values of the time constants of the system.

To illustrate the procedure in detail, let us consider the frequency response curve that corresponds to a transfer function with only real poles. That is to say, the factors of the transfer function are all assumed to lie in the denominator, and there are no quadratics. The frequency response curve of this function will show a continually

[5] The asymptotic curve for any single time-constant factor consists of only two straight lines, one asymptotic to the low ω end and the other to the high ω end of the exact curve. The term *asymptotic* is retained for the over-all curve obtained by taking the sum of the component curves, even though, in the over-all curve, only the first and last segments are asymptotic to the over-all exact curve.

[6] This correction curve is the same for all time-constant factors if the correction is taken as a function of ωT_k, where T_k is any time constant.

increasing slope, as in curve E of Fig. 1. Let us suppose that this curve, plotted ω db-log on scales, has been experimentally determined, and our problem is to determine the transfer function to which it corresponds. Such an experimental curve, equivalent to the "exact" curves of our previous discussion, is not made up of straight line segments and shows no directly perceptible features that identify the factors of the transfer function. If, however, we can derive from it the corresponding asymptotic curve, the corner points of the latter and the slopes of adjacent segments will provide us with the necessary information.

The procedure proposed may be regarded as a graphical method of successive approximation, carried out in a series of stages. As a preliminary step, the asymptotes are drawn to the low and high ω ends of the curve, using straight lines with slopes that are some multiple of 6 db per octave. At the low ω end this is likely to be a line with a slope equal to zero or – 6 per octave. The first step of stage one consists in finding a trial set of corner points. As a basis, it is assumed that the db correction (ε) at each corner point will be 3 db. This assumption represents a limiting condition, and is accurate only for a single time-constant factor or for transfer functions with widely separated corner points (e.g., at intervals greater than about 3 octaves). The assumption is useful, however, as a starting point in the series of successive approximations.

To locate the corner points, a pair of dividers is set at the 3-db separation, and moved along the low frequency asymptote, toward the higher frequencies, until the point on this line is found which is 3 db distant, along an ordinate, from the experimental curve. This point is the first trial corner point. A second straight line is drawn from this point, with an increment in slope of – 6 db per octave. The first point on this line that differs from the exact curve by 3 db gives the second corner point. The procedure is continued until one of the straight line segments intersects the high frequency asymptote, giving the final corner point. The total number of corner points associated with the curve should be equal to or greater than the difference between the slopes of the final and initial asymptotes, measured in db per octave, and divided by 6. (In a frequency curve with inflections, there are an additional number of corners equal at least to number of inflections.) It is frequently more effective, in locating the corner points, to work alternately from the low and high ω ends until an intersection of straight line segments occurs in about the middle of the frequency range.

This initial set of corner points provides an approximation to the density of the correct distribution of corner points. This initial set can now be used as a basis for obtaining a more accurate estimate of the db correction at each of the corners. This computation is carried out in the standard way described for the direct db-log frequency method, and indicated above (*2, 7, 8*). A table giving corrections for various values of $\omega/1/T$ or ωT can be used to accelerate the process.

The second stage of the approximation can now be carried out in a manner similar to that described. A second set of trial corner points is determined by using as db corrections (in the locating of successive corners) the values just computed instead of the initially assumed 3-db values. This new distribution of corner points is then used in the calculation of a second set of correction values. The procedure can be repeated until the shifts in location of the corner points with each new stage becomes less than some previously established criterion. A more basic test can be made by using the direct method to plot an exact "theoretical" curve from the final set of corner points, and comparing this theoretical curve with the experimentally given curve. If the agreement between the two is not considered satisfactory, the inverse process of successive approximation can be carried further.

The procedure described was adapted to a frequency curve with a progressively increasing slope up to the final asymptote. The result will be a transfer function containing only single order factors in the denominator. Some adjustment in procedure is needed for curves that may differ from this pattern. These modifications can be described briefly by reference to the factors in the transfer function to which they correspond.

In considering the significance of factors in the numerator of transfer functions, the following terms used in complex function theory will be found convenient. The zeros of a function are the roots of the factors in the numerator; the poles are the roots of factors in the denominator. A transfer function with a zero intermediate in frequency between that of two poles will be represented by a frequency diagram such as that of Fig. 2. Corresponding to a zero represented at ω equal to $1/T_2$ there is a reversal in the sign of the increment in slope of the following segment; corresponding to a pole at ω equal to $1/T_3$ there is another reversal in the sign of the slope increment. At frequencies just below those at which these reversals in sign take place, we note that the exact curve intersects the asymptotic curve. This relationship suggests the following rule for incorporation in the inverse method. Whenever a given segment of the trial asymptotic curve crosses the experimental curve, the following segment should be drawn with a reversal in the sign of the slope increment of 6 db per octave. The effect of the rule is to keep the asymptotic curve from deviating too far from the experimental curve.

The possibility of higher order factors is most easily handled by regarding each such factor as equivalent to a product of the appropriate number of single order factors. The presence of such higher order factors will therefore tend to give rise, in the inverse method, to the determination of corner points very close together. If this situation occurs, one may try the effect of substituting a second order or double corner point, as at B, for two single order corner points, as at A and C of Fig. 3. The manner of locating B by extending the previously found straight line segments until they intersect at B is indicated by the figure. The substitution of B for A and C eliminates segment AC as part of the asymptotic curve and implies a second-degree factor in the transfer function in place of two single-degree factors. The final test of whether the substitution is desirable will be the agreement obtained between the original experimental frequency curve and the theoretical frequency curve corresponding to a given set of corner points.

Finally, a word must be added concerning the procedure necessary for handling possible quadratic factors in the transfer function. A fuller discussion of means for experimentally detecting such factors in the experimental system is reserved for a later paper. One possible approach may be indicated. The presence of a quadratic factor can be detected by observing or recording the transient response of a given system to a step function. Quadratic factors, corresponding to conjugate complex roots, reveal themselves in the transient by damped oscillations. If these can be measured with sufficient accuracy, the damping ratio ζ, and the time constant T_q, can be identified (*2, 7, 8*). The specific db-log frequency curve corresponding to a quadratic with the given ζ and T_q can then be determined by computation, or taken from a chart of nondimensional quadratic frequency curves (*7*). The total curve corresponding to the quadratic can then be subtracted graphically from the over-all frequency response curve, leaving a residual curve devoid of the influence of quadratic factors. The inverse method described above may then be applied.[7]

Once the asymptotic curve has been determined, the process of writing the corresponding transfer function is relatively simple. The frequencies of the various corner points are written along the ω axis in the form $1/T_k$. One now proceeds from the low frequency end of the graph, writing down factors to correspond to successive segments. An initial segment at a slope of 6 m db per octave implies a $(j\omega)$ factor with an exponent equal to m. It is located in the denominator if the slope is negative and in the numerator if it is positive. A time constant factor $(j\omega T_k+1)^n$ is written for each successive corner point, with the appropriate value of T inserted in the factor. The degree of the factor equals n, where the increment in slope of the segment following the corner point is $6n$ db per octave. If the increment in slope is negative, the factor goes into the denominator; if positive it goes into the numerator. In case a quadratic wave-form has been initially subtracted from any part of the experimental frequency curve, a corresponding factor, $(j^2\omega^2T_q^2+j\omega 2\zeta T_q+1)$, is inserted in the transfer function, with numerical values for T_q and ζ. A function constructed in the manner described represents the frequency-dependent part of the total transfer function, and may be designated as the function $G(j\omega)$, in the terminology of servotheory. It can be converted into the more general function $G(s)$ by substituting the complex variable s for $j\omega$ in all factors of the transfer function (*4, 6, 8*).

[7] A general estimate of the accuracy attainable with the inverse method, for all types of experimental frequency curves, is not as yet available. For the frequency curves with which it has been tried thus far, curves corresponding to transfer functions with four or five poles and zeros, it was found possible to determine correctly the number and location of factors, in numerator or denominator, and the frequency coordinate of the corner points, with errors of the order of 10% after three or four stages of approximation.

The magnitude of the constant gain coefficient K can be determined by inspection of the initial (low ω) segment of the asymptotic curve (*7, 8*). Its height above the zero db axis (given in db units, and converted into units of gain), at ω equal to one, gives the value of K. The final transfer function, $KG(s)$, thus derived from the experimentally given frequency curve, will be in factored, nondimensional form. From this form it can be readily converted to nonfactored form by multiplying out the various factors; and can then be reformulated as a differential equation (*8*, p. 238). In either form, it can serve as a compact representation of the system, and as a basis for predicting its response to any arbitrary disturbance.

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Some New Finds of Fossil Ganoids in the Virginia Triassic[1]

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In the course of an extensive study of the carbonaceous and bituminous sediments of the Triassic basins in Virginia, a number of heretofore unrecorded fossil fish localities were discovered and investigated.

In October 1942 Martin examined a fresh roadcut along State Route 55 near Thoroughfare Gap in Prince William County and discovered a heretofore unknown or unmapped extension of the nearest Triassic basin (*1*). Along the base of the cut the imprint of a ganoid about 6 in. long, including most of the head and forward fins, was found in an exposure of very soft and

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