to us here. These entirely necessary expenses mean increased taxes and financial troubles for all, a condition that may continue throughout the rest of our lives.

Support of this great defense program is paramount and essential, but with it let us not forget that in cultural and esthetic pursuits there are not only improvement for the mind and training for the future, but also momentary escape for the individual from the troubles that beset him. In the halls of our museums, our art galleries and our libraries throughout our great nation there is found enjoyment and recreation for the public to be encountered nowhere else. The contemplation of nature and its laws, and of the individual objects that exemplify these, brings a relief and a peace not elsewhere possible. Public morale, of maximum importance under the grim threats of war, is fostered by such mental relaxation. These are facts to be remembered in periods of stress, that the small financial support for such activities be not denied. Let us consider this as a contribution to the defense armament of the mind and of the soul.

SOME UNSOLVED PROBLEMS OF THEORETICAL DYNAMICS[1]

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As was first realized about fifty years ago by the great French mathematician, Henri Poincaré, the study of dynamical systems (such as the solar system) leads directly to extraordinarily diverse and important mathematical problems in point-set theory, topology and the theory of functions of real variables.

On the other hand, the abstract point of view emphasized by the foremost American mathematician of the same period, E. H. Moore of the University of Chicago, led him in the early years of the present century to his "general analysis." Moore sought to introduce an absolutely general independent variable, ranging over an abstract space, whereas previously attention had been limited to an independent variable ranging over ordinary n-dimensional space. He hoped that in this way the abstract essence of various current theories in analysis might be more clearly revealed. Ideas of a somewhat similar type had been proposed a little earlier by Maurice Fréchet and also by Erhard Schmidt. But only Moore saw the full significance of general analysis for mathematical thought; and it is only in recent years that his ideas are receiving the attention which they deserve from mathematicians.

An early illustration of the wide scope of these Moorean ideas was furnished by the "recurrent motions" of dynamical systems first defined and studied by the writer in 1910, shortly after the completion of his graduate studies at Chicago. The possibility of making an extension of this theory so as to define "recurrent motions" and certain analogous "central motions" in the sense of general analysis was announced by him in his Chicago Colloquium Lectures on Dynamical Systems in 1920.

The principal part of his paper was occupied with this abstract phase of dynamics, which has been the subject of much recent work by American mathematicians and by the powerful contemporary Russian mathematical group. The kind of abstract space, R, which it seems best to employ is a compact, metric space. Corresponding to the change in "time" t there is a steady flow of the space R into itself, each point tracing out a "curve of motion" in R. The individual points represent "states of motion," and each curve of motion represents a complete motion of the abstract dynamical system. Thus there is provided not only an abstract space R but a "continuous group": $G: t' = t + c$. In other cases this group may be discrete: $t' = t + n$ (n, an integer), or of still more complicated form. For a continuous flow in such an abstract space R, the recurrent motions are merely those which trace out with uniform closeness in *any* sufficiently large period of their entire history, all their states; a periodic motion, represented by a closed streamline, affords the simplest illustration of such a recurrent motion. The analogous central motions are those which recur infinitely often near to any particular state of the motion, or at least have such motions in the infinitesimal vicinity of any state.

The first ten of the sixteen problems presented and briefly discussed were of this abstract type.

Problem 1 embodied a conjecture as to the interrelationship between continuous and discrete flows in such an abstract space R. It is easy to see that this relationship must be an intimate one by recalling the close connection between an ordinary changing visual image of continuous type and the corresponding moving-picture image of discrete type. In the abstract space R a species of reduction of a continuous flow to a discrete flow or at least one of "extensibly discrete" type may be effected by a process of sectioning,

[1] Summary of a paper presented at a fiftieth anniversary symposium of the University of Chicago, September 24, 1941.

first employed by Whitney in a local manner. It was conjectured that conversely any such extensibly discrete flow may be imbedded in an ordinary continuous flow. Ambrose and Kakutani have recently obtained interesting results lying in the same general direction as this first problem.

In problems 2 and 3 it was conjectured that *all* the motions of a continuous flow will be recurrent if and only if the flow may be decomposed into a set of irreducible constituent flows which are "homogeneous" (*i.e.,* such that the stream lines are topologically indistinguishable from one another). Thus the familiar two-body problem for a sufficiently small value of the energy constant is of this type, the irreducible constituents being the individual periodic elliptic motions.

The flows which arise from ordinary dynamical problems are not only continuous but in general are "conservative," *i.e.,* leave a volume integral invariant, as in the case of the flow of an incompressible fluid. This property of conservativeness was used about seventy-five years ago by Boltzmann and Maxwell in the foundation of statistical mechanics. It is easy and natural to extend the definition of conservative flows to the abstract case. Important studies of abstract conservative flows have been made recently by Beboutoff, Bogoliuboff, Kryloff, Stepanoff in Russia and by Halmos, Oxtoby, Ulam, von Neumann, Wiener and Wintner in this country, among others. The field of mathematics devoted to the study of conservative flows has risen to the rank of an important branch of mathematics, called "ergodic theory." This theory is destined to play a fundamental role in statistical mechanics, although as yet its importance for this field has not been generally realized by physicists.

Problem 4 was concerned with such conservative abstract flows. Here the interesting conjecture was advanced that at least if the abstract flow is so regular as to be "geodesic," then it will be conservative if all the motions are central. The converse fact was essentially established by Poincaré in the third volume of his "Méthodes Nouvelles de la Mécanique Céleste."

The reasonableness of this conjecture was based upon the use of a modified type of "compressibility volume" of the kind introduced by E. Hopf, and an analysis of recent remarkable results of Denjoy which established the unexpected fact in a simple special case that the *ultimate* behavior of a dynamical system may depend on the degree of regularity of the functions which characterize it.

In problem 5 it was likewise conjectured that the recurrent motions are necessarily everywhere densely distributed in the space R of a conservative flow. Poincaré has made an analogous but stronger conjecture in the case of the restricted problem of three bodies and of certain analogous problems when R is a three-dimensional space, namely, that the periodic motions are everywhere dense in the totality of motions, but it is known that his conjecture does not always hold. Questions of this general type are of philosophical interest, since the crude speculation that all dynamical systems are periodic or nearly so presents itself irresistibly to the human mind.

It was emphasized that from another point of view the real significance of the conservativeness of a flow is that (almost) all motions have *habitual modes of behavior in the mean with respect to any measurable process.* For example, consider the idealized frictionless motion of a billiard ball on a billiard table which has the shape of a convex oval. In any such motion the ball will be in the long run a definite proportion of the time on any assigned part of the table, will collide with the rim at a certain definite angular rate, etc. Problem 6 proposed a topological characterization of conservative flows based on this fact, similar to that given by Oxtoby and Ulam, in an as yet unpublished paper.

In problem 7 the restriction of continuity upon a conservative flow was relaxed, and a characterization of the invariants of the flow based on certain "packing coefficients" was proposed. A characterization of certain special types of such flows in terms of their "spectra" has been recently obtained by Halmos, von Neumann, Wiener and Wintner.

Up to this point continuous steady flows in R and the more special "conservative" type had alone been considered. But the continuity and conservativeness combined do not suffice to characterize the flow of true *dynamical* type except in the simplest case of two dimensions ($n = 2$). Hence it is of especial importance to define abstractly a "dynamical" flow. This was attempted by the writer. Roughly speaking, he takes Pfaffian systems as the model for his abstract definition rather than the more familiar but equivalent Hamiltonian systems of classical dynamics. In this way his task becomes that of formulating an abstract equivalent for the varitional condition,

$$\delta\int\Sigma X_i dx_i = 0.$$

The crucial part of his characterization of a dynamical flow lay in the suitable definition of a line integral in any abstract "geodesic space" R. One conspicuous advantage of such a characterization of a dynamical flow is that the flow in any invariant subspace of R is seen at once to be of dynamical type also.

It should be emphasized that hitherto the question of the adequate characterization of a dynamical flow beyond the obvious facts of continuity and conservativeness has been especially baffling. The proposed analytic characterization and the conjectured qualitative characterization embodied in problems 8, 9 and 10 should prove suggestive in this connection. In problem 8 it was asserted that a dynamical flow is necessarily conservative; in problems 9 and 10 that, certain cases aside, not only are the periodic motions everywhere dense but *the stable periodic motions are everywhere dense and dense on themselves.* Here a stable periodic motion was defined purely topologically as any periodic motion in whose infinitesimal vicinity lie other complete motions. A partial converse is known to hold through results obtained by D. C. Lewis and the lecturer.

Problem 11 was of a nature intermediate between the case of an abstract space R and a space R_n of n dimensions, and was the only problem not stated in complete form. It called for the appropriate generalization to a gas of certain remarkable results for the famous three-body problem due to Sundman, and extended by the writer and Hinrichsen to $n > 3$ bodies and to a more general law of force than that of Newton.

Problem 12 called for an example to show that in the case of a continuous (non-conservative) flow in a space R_n of $n \geqq 3$ dimensions, the ordinal series of "wandering motions" leading to the central motions need not always terminate in n or fewer steps.

In problem 13 it was conjectured that essentially the only 3-dimensional discrete flows which are "regular" in the sense of Kerékjártó are (1) combined rotations of three circles into themselves; (2) combined rotations of circle and surface of a 3-sphere into themselves; (3) rigid rotation of a 3-dimensional hypersphere into itself.

Problems 14 and 15 were closely related. The first of these asserts that a 1–1 direct analytic area-preserving deformation of the surface of a sphere into itself which has two fixpoints, and is such that iterates of the transformation have no other fixpoints, is a pure rotation from a topological point of view. Considerable evidence was adduced for this conjecture. The second problem embodied an analogous conjecture concerning a plane circular ring.

The last two of the announced problems (problems 16, 17) will perhaps excite the most interest, since they embody conjectures which in a certain sense yield a kind of complement to the famous "last geometric theorem" of Poincaré, announced as probably true by Poincaré shortly before his death and established subsequently by the lecturer. Suppose that there be given a ring-shaped part of the plane bounded by two concentric circles. Suppose that this ring is deformed into itself in any way so that the areas of small figures are conserved, while the points on the two circles are advanced by angular distances α and β. If α and β are distinct, Poincaré's theorem leads at once to the conclusion that there are infinitely many periodic sets of points under the indefinite repetition of this deformation. But if α and β are equal, his theorem is not applicable. The conjecture was made that the same result (as well as other more specific ones) will hold in the case $\alpha = \beta$, provided that some nearby points of the ring become separated widely in an angular sense by sufficient repetition of the deformation, as clearly happens when α and β are unequal. This conjecture was proved in the very important special case when the given conservative deformation can be expressed as the product of two involutoric deformations.

In consequence, for the classic restricted problem of three bodies treated by the American astronomer G. W. Hill, so long as there exists a "surface of section," either there exist infinitely many periodic motions (for a given value of the "constant of Jacobi") or all possible motions of the "infinitesimal body" (the Moon in the Earth, Moon, Sun case) will necessarily have the same mean rate of synodical advance of perigee about the near-by finite body (the Earth), per synodical revolution. It was also pointed out how the absence of infinitely many periodic orbits would indicate that a new *qualitative* integral exists, in addition to the usual analytic integral of Jacobi.

The problems presented and discussed by the writer will be likely to receive attentive consideration from other mathematicians inasmuch as they embody challenging conjectures concerning important open questions in the actively advancing field of theoretical dynamics.

OBITUARY

ELMER SAMUEL IMES

IN the death of Elmer Samuel Imes science loses a valuable physicist, an inspiring personality and a man cultured in many fields.

Professor Imes was born on October 12, 1883, in Memphis, Tennessee, the son of Home Missionary parents. His father, Benjamin A. Imes, a graduate of Oberlin College and Theological Seminary, was among the pioneers in educational and church work in the southern field of the American Missionary Association.

Imes taught for many years in the American Missionary schools, principally in Albany Normal School, Albany, Georgia, before he pursued his ultimate and highest interest, the field of physics and its educational and commercial adaptations.

He was graduated from Fisk University in 1903, and did graduate study both there and in the Uni-