as a theorem in molecular dynamics lies in the remarkable agreement between the results obtained by the methods described in the three different sections of this report, all of which are based on different fundamental hypotheses.''

EDWIN H. HALL.

CAMBRIDGE, October 28, 1899.

Elementi di calcolo infinitesimale con numerose applicazione geometriche. Per ERNESTO CESÀRO, professore ordinario della R. Universita di Napoli. Naples, Lorenzo Alvano. 1899. 8vo. Pp. 400.

The absence of a text-book on the calculus from a too well-known series of American mathematical text-books was recently remarked. The omission was excused by the observation that the author of the series knew nothing about the calculus. It might have been well for the cause of secondary and superior mathematical education in this country had the same modest confession been called into execution earlier and prevented the construction of the patch-work, fragmentary, stereotyped algebra of the same series. Contrast the confession of the razor-maker with the refusal made lately by a mathematician who declined to prepare an elementary treatise on the infinitesimal calculus on the ground that he knew too little arithmetic and algebra.

Cesàro had the courage to learn and make his mathematics before he began to publish any of his courses. His treatise* on algebraical analysis appeared five years ago and was most favorably received, although published against the advice of his friends. This work naturally contained an introduction to the infinitesimal calculus which gave full promise of the superb treatise which comes from the press this year. The former, which is by no means so finished a work of art as the latter, is a collection of sixty lectures on substitutions and determinants, linear forms, quadratic forms, irrational numbers, limits, series, functions, developments in series, complex numbers, quaternions, elimination, symmetric functions, enumeration of roots, numeric and algebraic resolution of equations, differences and interpolation, and factorial developments.

*Cesàro, Corso di Analisi algebrica con introduzione al Calcolo infinitesimale, Turin, Bocca, 1894.

Cesàro's course in the calculus is designed after the following plan the style of whose exposition is a most fortunate combination of mathematical rigor and poetic expression. There are three grand divisions occupied in order with fundamental theories, the differential calculus, and the integral calculus. The first of these consists of four chapters devoted to functions, derivatives, developments in series, and functions of several variables; the second part also contains four chapters presenting the theory of differentiation and its applications to the theories of plane curves, space curves and surfaces; the last division comprises five chapters on integration, applications to the evaluation of certain remarkable classes of integrals, applications to geometrical mensuration, differential equations and variations.

The reviewer has space to analyze but few of the chapters of this valuable work. The first chapter exhibits the principal properties of functions in all their modern refinement by the evolution of the following theorems: 1° If a function is finite throughout an interval it always admits of an inferior limit and a superior limit; 2° If a function is finite for all the numbers of an interval it is finite throughout the interval; 3° The first theorem of Weierstrass, if a function is finite in a finite interval, the latter contains at least one number for which the function has the same limits, inferior and superior, as the interval itself; 4° For the existence of a finite limit of $f(x)$ to the right of a it is sufficient that, given ε positive and as small as we wish, there can always be found a positive number h, such that, for every pair of values x' and x'' taken within the interval $(a, a+h)$, excluding the inferior limit, the absolute value of $f(x') - f(x'')$ is less than ε; 5° If $f(x)$ is continuous and different from zero for $x = a$, it possesses at a the sign of $f(a)$; 6° If a function is continuous in an interval it is also finite in the interval; 7° A function continuous in an interval at the extremities of which it takes opposite signs must vanish at least once in the interval; 8° A continuous function cannot pass from one value to another without passing through all the intermediate values; 9° Second theorem of Weierstrass, every function continuous in a finite interval takes the maximum and minimum value in the same interval; 10° Cantor's theorem, if a function is continuous in a finite interval, we can determine for every positive number ε as small as we wish, a number h, such that in any interval of magnitude h contained within the given interval, the oscillation of the function shall be less than ε.

The second chapter deduces by the method of limits the rules of derivation of standard functions, together with the properties of derived functions, and concludes with the complements of the theory of limits introduced in the first chapter. The third chapter devoted to series discusses in order the convergence criteria, the Taylor-Maclaurin formula, the asymptotic evaluation of power-series, the technical discussion of functions, the interpolation formula and the decomposition of rational functions into sums of simple fractions. The notions of the first chapter are extended to functions of more than one variable in the fourth chapter, with special reference to the problems of maxima and minima. The examples and exercises of these chapters, most of which are resolved in full, are especially valuable; the collection of classic ones of derivativeless functions calls for remark; the character of these exercises is well exemplified by the following which are given in illustration of the theory of maxima and minima: 1° Calculate the lengths of the axes of the general conic; 2° Determine the lengths of the axes of the section of an ellipsoid made by a given diametral plane; 3° Find the minimum distance between two right lines; 4° Seek the minimum value of the sum of the squares of n variables connected by $m < n$ linear equations; 5° The method of least squares.

In this day of multiple algebras and multiple geometries it is not surprising to find Cesàro proposing multiplications of the differential calculus. These observations form an interesting section of the fifth chapter which gives the ordinary methods for the differentiation of explicit and implicit functions of one or several variables. The differential dx of the independant variable, arbitrary for each value of x, Cesàro considers as the product of an infinitesimal a independent of x by an arbitrary function of x, *i. e.*, $dx = a\chi(x)$. Differentiating this expression we have

$$d^2x = da\chi = ad\chi = a\chi' dx = a^2\chi\chi',$$
$$d^3x = a^3(\chi\chi'^2 + \chi^2\chi''),$$
$$d^4x = a^4(\chi\chi'^3 + 4\chi^2\chi'\chi'' + \chi^3\chi'''), \cdots$$

The results of these successive differentiations become rapidly more complicated, and would as rapidly rob the calculus of most of its advantages if the function χ be allowed to retain its arbitrary character. For convenience $\chi(x)$ is made equal unity and we have $d^2x = d^3x = \cdots = 0$, which expresses that x is equicrescent, *i. e.*, that the differential of the independent variable is independent of the variable. However it is only necessary to call in the fundamental principle of the integral calculus to show that every form of calculus resulting from a change of form in the function χ reduces to the ordinary calculus; the reduction is effected in precisely the same manner that a change of independent variable is made. Thus, there is always a function t of x whose derivative is $1 : \chi(x)$, then

$$dt = t' dx = \frac{1}{\chi(x)} \cdot a\chi(x) = a, \quad d^2t = d^3t = \cdots = 0.$$

The possibility of a calculus in which no variable possesses a constant differential is not excluded, but it is certain that the simplicity and homogeneity of its formulæ and the precision with which the ordinary calculus assigns the orders of its infinitesimals will not be among the advantages of the new calculus.

It may be remarked here in passing that a Norwegian mathematician attempted a few years ago to found a new calculus, in which the fundamental rôle taken by addition and subtraction in the ordinary calculus was assigned to the operations of multiplication and division. The resulting forms yielded certain continued products, but were otherwise fruitless.

The sixth and seventh chapters contain the geometrical applications to plane and space curves. These chapters must have offered a sore temptatation to the author to make exclusive use of his own elegant method of intrinsic analysis, but the reader finds no method employed to the exclusion of all others. The applications follow the usual order of tangents, normals, curvature, asymptotes, singularities, contacts, and envelopes. The examples are happily chosen, and the chapters amply illustrated with well executed figures.

The elements of the theory of surfaces are introduced in the eighth chapter and applied to ruled surfaces and envelopes. The theory of curvature is elaborated in detail, including the notions of mean curvature due to Germain, total curvature conceived by Gauss, and quadratic curvature of Casorati. The chapter concludes with the determination and properties of the remarkable lines of a surface.

The ninth chapter begins the study of the inverse problems by presenting the fundamental concepts and rules of simple and multiple integration. The tenth chapter evaluates the well-known forms of rational, irrational and transcendental indefinite integrals, and terminates with certain classes of definite integrals, including elliptic and eulerian integrals; the nature of the xample is indicated by the following, which occurs in the study of vortices:

$$\frac{ab}{2\pi}\int^{2\pi}\frac{\cos\theta d\theta}{\sqrt{a^2+b^2+c^2-2ab\cos\theta}}.$$

After making the ordinary applications to mensuration in the eleventh chapter, the author undertakes the elements of the theory of differential equations in the twelfth chapter. The distinctions between the notions general, particular, and singular integral are clearly made. The cases of integrable ordinary differential equations are classified as follows: 1° variables separable; 2° functions homogeneous; 3° one variable absent; 4° second order equation lacking one variable always reducible to one of first order; 5° linear equation; 6° Bernouilli's equation; 7° Clairaut's equation; 8° the form $y=x\phi(y')+\psi(y')$, when not a Clairaut equation is reducible to a linear equation; 9° Riccati's equation and its characteristic property that the anharmonic ratio of any four particular integrals is constant. No reference is made to Lie's theories. A well selected list of resolved problems is followed by geometrical applications of differential equations to plane curves, trajectories and surfaces. The general linear equation and equations with constant coefficients are studied somewhat *in extenso*. Passing then to equations in more than two variables, the author takes up total differential equations and simultaneous ordinary equations and terminates the chapter with a short treatment of the partial differential equation.

The last chapter of the book gives the elementary notions of the calculus of variations in six pages. The volume concludes with notes on the concept of limit, oscillatory extremes, demonstration of Cantor's theorem, Hadamard's theorem, minima and maxima of functions, cusps and flexions at a pole, torsion of curves, calculation of the curvature of a surface, formulæ of Rodrigues, general formula of Stirling.

E. O. LOVETT.

PRINCETON, NEW JERSEY.

Pflanzen- und Tierverbreitung, in Hann, Hochstetter und Pokomy, Algemeine Erdkunde. By A. KIRCHHOFF. Verlag, F. Tempsky, Wien. Aufl. 5. 1899.

This volume, by Alfred Kirchhoff, forms the third part of the new edition of a well-known and compendious manual of pure as distinguished from economic geography. It maintains the high standard of the preceding parts by Hann and by Brückner, and is a welcome addition to the literature of geo-biology. Of the 157 figures, a large proportion are not easily accessible elsewhere or are quite new. The maps, while not emphasizing the developmental phases of faunal and floral distribution as do, for example, those of Engler, are, nevertheless, more nearly in accord with modern ideas than those of Grisebach or Decandolle. The ecological factors are, by no means, neglected, as they were so generally in the older books. While it is true that they are scarcely so exhaustively discussed and laboriously analyzed as in the special treatises of Warming and Schimper, yet they are clearly, ably and adequately presented. Kirchhoff's work, has a certain advantage over the special *Tierlebens* and *Pflanzen-geographies* in its broad outlook upon both the fields of biological science. It falls naturally enough into three divisions, the first including the general discussion of the relations between the earth and the organisms that inhabit it, the second comprising the analysis of floral, and the third that of faunal regions. The peculiar excellence of the treatment is apparent at once in the opening chapters on the migrations of organisms, on the environmental conditions