PORTRAITS OF SCIENCE

Proof, Amazement, and the Unexpected

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Archimedes is famous for his "Eureka" moment and for his war machines, but the subtle mathematical proofs in his treatises are his most enduring legacy.

Reviel Netz

Archimedes shows up in the most unexpected places: it is possible he is mentioned in the Bible. Ecclesiastes (9:14–16) says "There was a little city, and few men within it; and there came a great king against it, and besieged it, and built great bulwarks against it: Now there was found in it a poor wise man, and he by his wisdom delivered the city; yet no man remembered that same poor man. ... The poor man's wisdom is despised." A century ago, Moriz Friedländer pointed out that this might be a version of the story of Archimedes in the siege of Syracuse—the simple citizen who humbled a great power only to be killed by a common soldier (*1*). Biblical scholars today would doubt his interpretation because Ecclesiastes is just telling us a story with a moral, but Friedländer did have a point. Archimedes indeed captured the public imagination of the ancient Mediterranean in a way no other scientist did.

The only certain date we have for Archimedes is his death in 212 B.C. as a victim of the second Punic War, the great World War of antiquity. Archimedes designed the clever defensive machines (including catapults that fired logs at the attackers) that allowed Syracuse to resist the Roman siege for 2 years. His inventions became a powerful symbol of how Greek wisdom could outwit Roman power. Despite the Roman general Marcellus' desire to save him, presumably because he wanted Archimedes' talents for Rome's benefit, Archimedes lost his life when the city fell.

And so Archimedes became the stuff of legend. We still hear more about him than about any other ancient scientist, although much of what is retold is unreliable. Nevertheless, valuable personal information occasionally occurs in his treatises. For example, in *The Sand-Reckoner,* Archimedes notes a result reported by an astronomer called Phidias, who—Archimedes mentions in passing—was his own father. The name is significant, because it shows that Archimedes did not hail from the aristocracy. The great sculptor of the Parthenon in Athens was named Phidias and since then, the name was attached almost exclusively to artisans. Craftsmanship was little valued by the ancient elite, and any manual work was despised, such that members of the elite never gave their sons names that smacked of artisanal achievement. Thus, Archimedes' grandfather was, very likely, not an aristocrat but a humble artisan (*2*).

Although the legends are unreliable, a discussion of Archimedes is incomplete without a visit to the baths. The most famous version of the story of King Hiero's golden crown was told by Vitruvius (*3*). During a visit to the baths, Archimedes is lost in thought contemplating the problem of how to test the purity of the gold in the crown without destroying it. Then Archimedes notices the water overflowing from his bath and immediately runs out crying, "Eureka, eureka!"

Archimedes

(ca. 287 B.C.—212 B.C.)

Archimedes was born about 287 B.C. in Syracuse on the island of Sicily. He died in 212 B.C. by the hand of a soldier when Syracuse fell to the Romans.

What had he found? According to Vitruvius, Archimedes had realized that the volume of water displaced by an object immersed in it is equal to the volume of the object itself. So if Archimedes put the crown in a bath of water and measured the displaced volume of water, this would be the same as the volume of the crown. The volume of the water displaced by the crown when compared with the volume of the water displaced by a similar mass of pure gold, as originally supplied to the goldsmith, should have been the same—but the volume displaced by the crown turned out to be greater. This meant that the King's goldsmith had stolen some of the pure gold, and had replaced it with a less-dense base metal when he smelted the metal for the crown.

The method, as recounted by legend, is sound, but it is based on a trivial observation, so trivial that it is not mentioned in Archimedes' treatise *On Floating Bodies*.

The bath anecdote does not give us the true measure of the man. In *On Floating Bodies*, Archimedes made the following, astonishingly subtle deduction: In a stable body of liquid, each column of equal volume must have equal weight; otherwise, liquid would flow from the heavier to the lighter. The same must hold true even if some solid body is immersed in such a column of liquid. In other words, if we have a column of liquid with a solid body immersed in it, the aggregate weight of the liquid and the body must be equal to that of a column of liquid of the same total volume. It follows that the immersed body must lose weight: it must lose a weight equal to the weight of the volume of water it has displaced. (This is why we feel lighter in the bath.) This fundamental theorem was proved by Archimedes, with perfect rigor, in *On Floating Bodies*, Proposition 7 (*4*). Now that's something to cry "eureka" about.

Austere and technical as they are, Archimedes' treatises are just as striking as the anecdotes about him. In the treatises three motives run together: proof, amazement, and the juxtaposition of the unexpected. Proof and amazement are related, because Archimedes amazes us by proving that something very surprising is in fact true. Amazement and the juxtaposition of the unexpected are related, because the amazing result is usually seen in the equality or equivalence of two seemingly separate domains.

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Archimedes very rarely makes arguments that merely appear intuitive—and, crucially, when he does, he says so explicitly. He sets out as postulates some very subtle assumptions. For instance, in the introduction to the *First Book on Sphere and Cylinder*, Archimedes asserts that if two lines are concave to the same direction, and one encloses the other, the enclosing line is greater than the enclosed and so, for instance, the line is the shortest distance between two points. He took enormous care to distinguish what can be proved from what cannot. By turning seemingly obvious observations into explicit postulates, Archimedes was able to set out truly incontrovertible proofs.

Archimedes' proofs always reached surprising, counterintuitive results. A simple jeu d'esprit serves as good example. In *The Sand-Reckoner*, Archimedes shows that his numerical system would allow the number of grains of sand that it takes to fill the entire universe to be counted. This proof served no obvious mathematical purpose—it was an exercise in amazement.

Typically, the outcome of his proofs took one of two forms. First, that a curvilinear object can be shown to be equal to some rectilinear object (the boundary between the curved and the straight is the heart of Greek geometry and indeed of geometry in general). Second, that physical objects can be described in abstract geometrical terms.

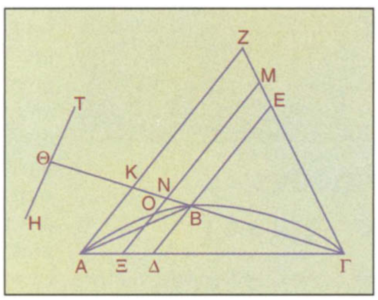
The treatises containing proofs on combinations of curved and straight geometries include the two books on *Sphere and Cylinder*. The configuration of sphere inscribed in a cylinder was so striking that Archimedes chose it to be inscribed on his tomb. Centuries later, this device might have been the inspiration for grander edifices, such as Constantinople's Hagia Sophia. No less striking are two treatises that brought descriptions of two new objects into the world: *Conoids and Spheroids*, and *Spiral Lines*. The treatises that describe physical objects in geometrical terms are the two books on *Planes in Equilibrium* and the two books *On Floating Bodies*. In two treatises, *Quadrature of Parabola* and the *Method of Mechanical Theorems*, both approaches are used simultaneously.

These, together with the *Sand-Reckoner*, constitute what is almost universally agreed to have been extant from Archimedes' own hand. It is perhaps the most influential body of work in the history of mathematics. If the process of proof, amazement, and juxtaposition of the unexpected may appear to present-day mathematicians as a fair description of what they themselves aim to achieve, then this may be due to the historical legacy of Archimedes, as transmitted through Galileo, Leibniz, and Newton.

Archimedes' treatise on the *Method of Mechanical Theorems*, which itself tends to turn up in unexpected places, was his most remarkable work. It was lost until the great philologist Heiberg discovered it in a palimpsest (a scraped and overwritten parchment) in Istanbul in 1906. Heiberg had discovered a 10th-century copy of the treatise, which had been used as the fabric for a 13th-century prayerbook. Heiberg was able to read much, but not all of the faint traces (*5*). Shortly after this astonishing discovery, the manuscript was lost or stolen, but in 1998 it resurfaced at a Christie's auction sale at New York. It sold for two million dollars. The anonymous owner generously supports the conservation and imaging now taking place at the Walters Art Museum, Baltimore (*6*).

"he by his wisdom delivered the city"

As we should expect of Archimedes, the results of our recent research on the palimpsest are indeed unexpected.



The schematic configuration illustrating the first proposition of Archimedes' *Method*. (Adapted from a tracing of the Archimedes Palimpsest prepared by William Noel, manuscript curator at the Walters Art Gallery, Baltimore, Maryland. My thanks go to him and to the owner of the manuscript for permission to reproduce the drawing.)

Since 1906, it has been known that in the *Method of Mechanical Theorems*, Archimedes combined concepts of straight, curved, physical, and geometrical. Above all, anticipating the calculus, he combined finite and infinite.

Take two objects, one curved and one straight. Divide each into infinitely many sections in such a way that, taken pair-wise, they all balance around the same single point (see the figure, this page). For instance, in the figure for the First Proposition, for any of infinitely many such sections, the two lines ΟΞ from the parabola ΑΒΓ and ΜΞ from the triangle ΑΖΓ (with the parabola-line positioned at the point Θ) balance around the point K. It follows that, taken as a whole, the triangle and the parabola also balance around the same point K (with the parabola's center of gravity at Θ). The center of gravity of a triangle is easy to find, so that we can now measure the distance between the centers of gravity of the parabola, and of the triangle, from the point around which they balance. In other words, we have measured the parabola.

So much we have known for a century. In a visit to Baltimore in 2001, Ken Saito from Osaka Prefecture University and I examined a hitherto unread piece of the *Method of Mechanical Theorems*. We could hardly believe our eyes: It turned out that Archimedes was looking for rigorous ways of establishing the calculus (*7*).

Modern scholarship always assumed that mathematics has undergone a fundamental conceptual shift during the Scientific Revolution in the 16th century. It has always been thought that modern mathematicians were the first to be able to handle infinitely large sets, and that this was something the Greek mathematicians never attempted to do. But in the palimpsest we found Archimedes doing just that. He compared two infinitely large sets and stated that they have an equal number of members. No other extant source for Greek mathematics has that.

This finding embodies the essence of Archimedes' lifework. Above all, he was trying to do what others before him had not done: to achieve the unexpected.

References

1. M. Friedländer, *Griechische Philosophie im altem Testament* (G. Reimer, Berlin,1904), pp. 151–157.
2. R. Netz, in *Science and Mathematics in Greek Culture*, T. Rihll and C. Tuplin, Eds. (Oxford Univ. Press, Oxford, 2002), chap. 11.
3. Vitruvius, *Architecture*, introduction to book IX, transl. F. Granger (Harvard Univ. Press, Cambridge, MA, 1934), p. 202.
4. Another anecdote about Archimedes, however, provides a method for solving the crown problem using the true discoveries of Archimedes in *On Floating Bodies*. Measure the weight of gold immersed in water and outside it; do the same with the crown; if the difference in weights is not the same, the crown is not made of pure gold! This alternative method derives from a 5th-century A.D. didactic poem in Latin about weights and measures: *Carmen de Ponderibus*, see http://www.fh-augsburg.de/~harsch/Chronologia/Lspost05/Remmius/rem_carm.html.
5. The manuscript also includes several other works by Archimedes, although almost all of them are known from elsewhere.
6. See http://www.thewalters.org/archimedes/frame.html.
7. R. Netz, K. Saito, N. Tchernetska, (part 1), *Sciamvs: Sources Comment. Exact Sci.* **2**, 9 (2001), and (part 2), *Sciamvs: Sources Comment. Exact Sci.* **3**,109 (2002).