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Graphical Evolution of the Arnold Web: From Order to Chaos

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We represent graphically the evolution of the set of resonances of a quasi-integrable dynamical system, the so-called Arnold web, whose structure is crucial for the stability properties of the system. The basis of our representation is the use of an original numerical method, whose definition is directly related to the dynamics of orbits, and the careful choice of a model system. We also show the transition from the Nekhoroshev stability regime to the Chirikov diffusive one, which is a generic nontrivial phenomenon occurring in many physical processes, such as slow chaotic transport in the asteroid belt and beam-beam interaction.

The long-term behavior of a mechanical system is in general unpredictable. In the frame-work of Hamiltonian systems, an exception is systems that are integrable in the sense of Liouville-Arnold. In these systems, the phase space is completely filled with invariant tori, and on each invariant torus all motions are quasi-periodic with the same frequencies $\omega_1, \ldots, \omega_n$, where n is the number of degrees of freedom. Although Liouville-Arnold's integrability is a rare property, many mechanical systems of great interest are integrable, such as the Euler-Poinsot rigid body, the two-body problem, and the Birkhoff normal forms around elliptic equilibria truncated at suitable order.

Many interesting problems of physics, such as the stability of planets and asteroids, of planetary spin-axis, and of the beam-beam interaction, among others, can be represented as small perturbations of integrable systems. In general, a small perturbation breaks the integrability of the system. Consequently, the behavior of the solutions can become complex and unpredictable to such an extent that it is generically called chaotic. Small perturbations of integrable systems transform them into quasi-integrable systems, and their study is the subject of Hamiltonian perturbation theory. One of the most celebrated results of Hamiltonian perturbation theory is the KAM theorem (*1–3*), which applies if the perturbation is smooth (*4*) and suitably small (*5*) and if the integrable approximation of the system satisfies a nondegeneracy property (*1, 3, 6–9*). The KAM theorem establishes that for the majority of initial conditions, which we call the regularity set, the features of the motions of the system are essentially those of the integrable approximation: In the regularity set, motions occur on invariant tori, and on the same torus all motions are quasi-periodic with the same frequencies. More precisely, the KAM theorem proves that for any invariant torus of the original system with nonresonant frequencies [more precisely, Diophantine (*10*)], there exists an invariant torus in the regularity

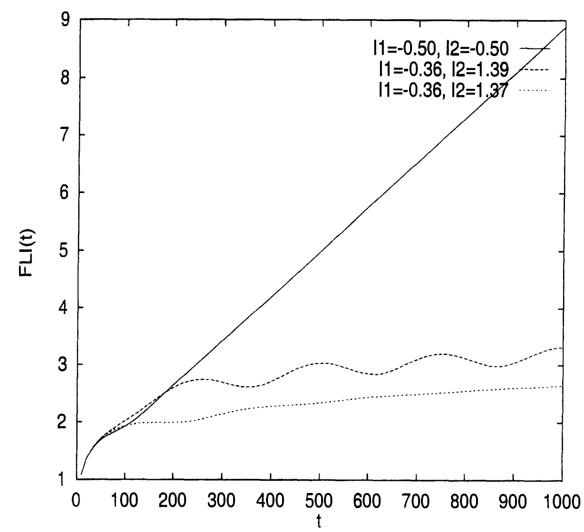

Fig. 1. Variation of the FLI for $\varepsilon = 0.01$ as a function of the integration time for a chaotic orbit (continuous line), a nonresonant one (dashed line), and a resonant one (dotted line). The three kinds of orbits are clearly distinguished already for $t = 300$.

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set that is a small deformation of the unperturbed one. Conversely, nothing is predicted by KAM theory for initial conditions in the neighborhood of the set made of invariant unperturbed tori with frequencies that satisfy a resonance condition $\Sigma_i k_i \omega_i = 0$ with some integers $(k_1, \ldots, k_n) \in \mathbf{Z}^n \backslash 0$, within a suitable accuracy that increases with the order (11) $\Sigma_i |k_i|$. Therefore, in the neighborhood of such a set, which is called the Arnold web, the motions of the system can exhibit chaotic features.

The topology of the Arnold web is peculiar. To describe it, we resort to the frequency space $\omega_1, \ldots, \omega_n$. In this space, the Arnold web projects on the frequencies satisfying $\Sigma_i k_i \omega_i = 0$ with a neighborhood that decreases with the order $\Sigma_i |k_i|$. Therefore, it is open and dense, and if the perturbation is suitably small, it has a small relative measure. This structure was explained analytically in (3) but only for very restrictive conditions (especially on the magnitude of the perturbation). In addition, the rigorous proof of the existence of instability and irregularity in the Arnold web is a delicate, not completely solved problem. For physically interesting systems, recent successful approaches are based on numerical investigations (12). In different fields of physics, the question of the stability of quasi-integrable Hamiltonian systems in the sense of the KAM theorem is important, because for the majority of initial conditions it provides stability for infinite times and describes motions. In beam-beam interactions (13), there is the problem of having to remain as close as possible to given computed orbits in order to indeed have interaction between particles. Within the old and not-yet-solved problem of the stability of the solar system, it is not completely clear whether the orbits of some planets (14) and of a significant number of asteroids (15) will change or not in an important way. Previous work has been based on numerical applications of the frequency-map analysis (16). Here we give a graphical representation of the Arnold web, obtained with a numerical test of regularity of the solutions of the system, with a sharpness never seen before.

We consider a system with the following Hamilton function

$$H_\varepsilon = \frac{I_1^2}{2} + \frac{I_2^2}{2} + I_3 + \varepsilon\left(\frac{1}{\cos\phi_1 + \cos\phi_2 + \cos\phi_3 + 4}\right) \qquad (1)$$

where $I_1, I_2, I_3 \in \mathbf{R}$ and $\phi_1, \phi_2, \phi_3 \in \mathbf{S}$ are canonically conjugated (17), and ε is a parameter that the larger it is, the more perturbed the problem becomes. The canonical equations of the integrable Hamiltonian H_0 are integrated: I_1, I_2, I_3 stay constant while the angles at time t $\phi_1(t) = \phi_1(0) + I_1 t$, $\phi_2(t) = \phi_2(0) + I_2 t$, $\phi_3(t) = \phi_3(0) + t$ rotate with constant angular velocity. Therefore, each couple of actions I_1, I_2 characterizes an invariant torus $\mathbf{T}^3$, and all motions on the considered torus are quasi-periodic with frequencies $\omega_1 = I_1$, $\omega_2 = I_2$, $\omega_3 = 1$. Conversely, for any small ε different from zero, H_ε is not expected to be integrable. However, we expect that the KAM theorem applies, and consequently the phase space is filled by a large

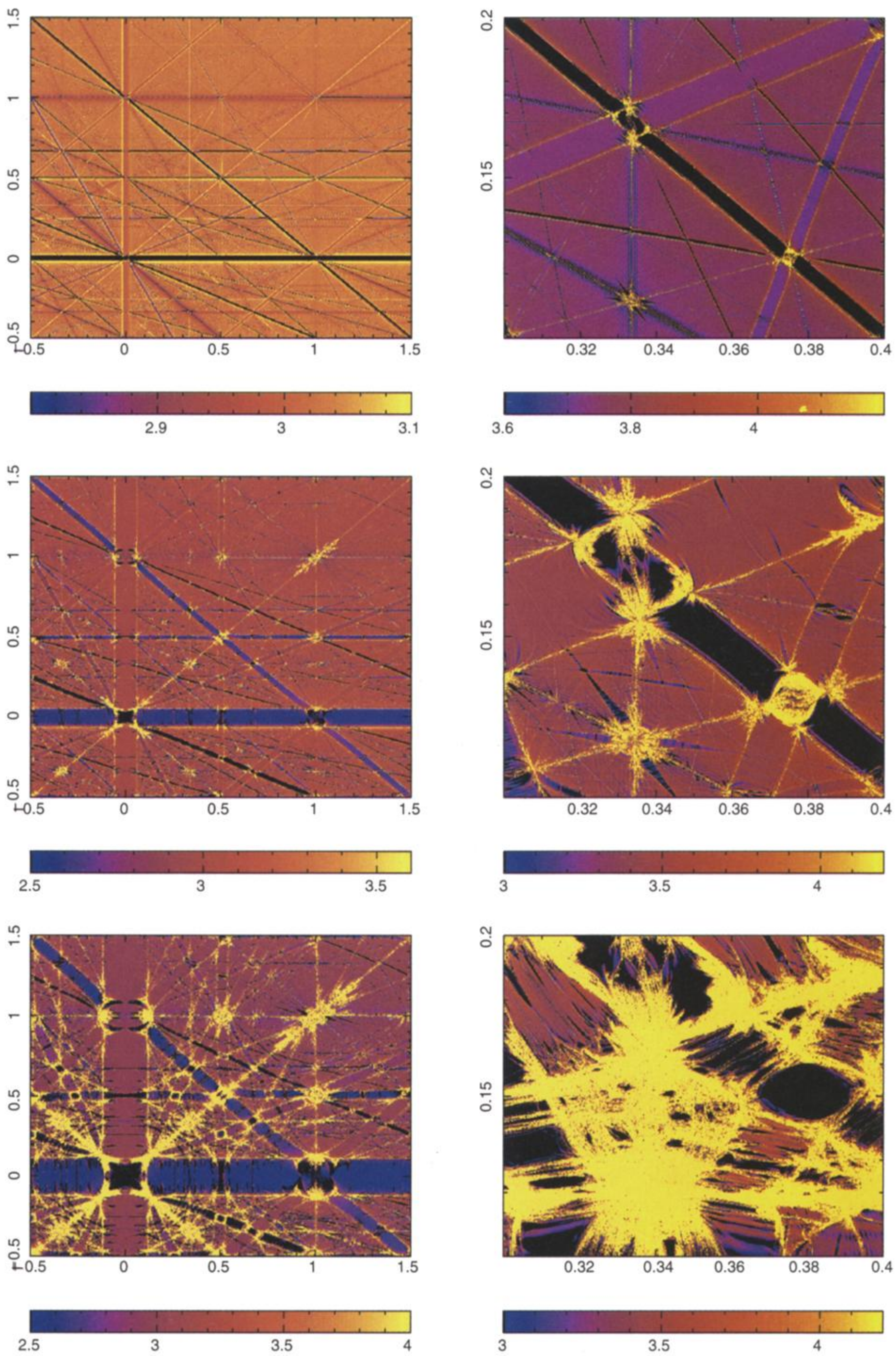

Fig. 2. Evolution of the Arnold web for increasing values of the perturbation parameter. The lowest values of the FLI appear in black and they correspond to the resonant islands of the Arnold web; the highest values appear in yellow and they correspond either to chaotic motion rising at the crossing nodes of resonant lines or to the presence of separatrix. The FLIs of all the KAM tori have about the same value, and therefore they appear with the same purple color. The choice of the color scale is suited to the value of the perturbation parameter and to the integration time. **(Left column)** A large portion of the action plane. Top: $\varepsilon = 0.001$, $t = 1000$; middle: $\varepsilon = 0.01$, $t = 1000$; bottom: $\varepsilon = 0.04$, $t = 1000$. **(Right column)** Enlargement of the figures on the left obtained with a large integration time in order to see smaller details. Top: $\varepsilon = 0.001$, $t = 4000$; middle: $\varepsilon = 0.01$, $t = 2000$; bottom: $\varepsilon = 0.04$, $t = 2000$.

volume of invariant tori, surrounded by the Arnold web. Our goal is to detect numerically the structure of the Arnold web. The Arnold web can be represented in the two-dimensional plane I_1, I_2, where each point individuates in a universal way the frequency of an unperturbed torus. Moreover, all resonances $k_1\omega_1 + k_2\omega_2 + k_3\omega_3 = 0$ are represented by straight lines $k_1I_1 + k_2I_2 + k_3 = 0$. Of course, the set of all resonances is dense on the action plane I_1, I_2. However, one can expect that irregular orbits surround each resonance line up to a distance that decreases as $\sqrt{\varepsilon}\exp(-\Sigma_i|k_i|)$, and consequently the volume of the Arnold web is expected to be as small as $\sqrt{\varepsilon}$.

We now describe the expected phenomenology of the motions with initial conditions in the Arnold web. Within resonances, both chaotic and regular motions can be observed. Regular resonant motions are topologically different from the regular nonresonant ones. These islands of regular resonant orbits can be surrounded by chaotic zones. However, orbits with initial conditions in such chaotic regions do not diffuse in the action plane because of the Nekhoroshev theorem (*18, 19*), which applies if ε is small (*20, 21*) and some nondegeneracy condition is satisfied (in particular, satisfied by the Hamiltonian in Eq. 1). By increasing ε one reaches a threshold ε_0 for which the global volume of resonances does not leave any place for invariant tori. In this case, the dynamics is no longer controlled by the Nekhoroshev theorem. To describe it, we use the Chirikov overlapping criterion (*22*), which allows the resonant chaotic solutions to go from one resonance to the other, possibly giving rise to large-scale diffusion. As a global picture, all the action space seems to be constituted by a large-volume chaotic region with some robust resonant island in it. Actually, the numerical approaches, including the one we use, allow estimation of ε_0.

To discriminate between chaotic and ordered orbits, it is usual to compute the largest Lyapunov exponent. A faster indicator is the fast Lyapunov indicator, hereafter called FLI (*23, 24*). Here, the new point is the use of the FLI to discriminate also between KAM tori and regular resonant motion, which is not possible with the Lyapunov exponents. Given a set of differential equations

$$\frac{d}{dt}X = F(X) \qquad X = (x_1, x_2 \ldots, x_n) \qquad (2)$$

under some regularity conditions (*25*), the Lyapunov exponents are computed by integrating the equations of motion and the variational equations $\frac{d\vec{v}}{dt} = \left(\frac{\partial F}{\partial X}\right)\vec{v}$, where $\vec{v}$ is any n-dimensional vector. The largest Lyapunov exponent is defined in such a way that, unless $\vec{v}(0)$ belongs to some lower dimensional linear spaces, the quantity $\ln\|\vec{v}(t)\|/t$ tends toward it for t going to infinity. If Eq. 2 is Hamiltonian and if the motion is regular, then the largest Lyapunov exponent is zero, otherwise its value is positive. The FLI is the value of $\ln\|\vec{v}(t)\|$ at fixed time t. This quantity, which can be easily computed with any set of coordinates, keeps trace of the topological differences between resonant regular motion and KAM tori (Fig. 1). Instead, $\ln\|\vec{v}(t)\|/t$ in the limit of infinite t goes to zero in both cases.

In recent years, other tools of analysis have been introduced, such as the frequency map analysis, the sup-map analysis, and the twist angle (*12, 16, 26–29*). We computed the FLI, using a leapfrog symplectic integrator, on a grid of 500×500 mesh of initial conditions regularly spaced in the action space (the choice of initial angles was $\phi_1 = \phi_2 = \phi_3 = 0$) with a fixed initial tangent vector $[v_{I_1} = 1, v_{I_2} = 1, v_{I_3} = 1, v_{\phi_1} = 0.5(\sqrt{5} - 1), v_{\phi_2} = 1, v_{\phi_3} = 1]$.

In Fig. 2, top left, and the enlargement shown in Fig. 2, top right, the resonant lines are embedded in large zones filled with KAM tori. Because of the choice of the perturbation with a full Fourier spectrum (that is, all harmonics are present at order ε), a large number of resonances are visible at small ε (in principle, all resonances should appear just by increasing the integration time). For $\varepsilon = 0.01$, the volume of invariant tori decreases and the chaotic regions become evident at the crossing of resonances, but the system is still in the Nekhoroshev regime (Fig. 2, middle left and right). For $\varepsilon = 0.04$, the dynamical regime has changed (Fig. 2, bottom left and right). The majority of invariant tori have disappeared because of resonance overlapping, and a chaotically connected region has replaced the regularity set.

We have shown that for very slightly perturbed systems, the Arnold web seems to indeed have the described structure, whereas when the strength of the perturbation is increased, the regular set shrinks until it almost completely disappears. In this way, the evolution from a mostly ordered system to a largely chaotic one is clearly represented, and it turns out to be in complete agreement with theoretical representations, except for the value of ε_0, which numerically appears in the range [0.01, 0.04]. The sharpness of Fig. 2 is not only important for didactic purpose but also gives hope that the method can provide a deeper comprehension of the more complicated physical systems.

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