MATHEMATICS

Is That Your Final Equation?

The Clay Institute announces a list of mathematical puzzles for the 21st century—at $1 million a pop

For decades, the fugitives have eluded capture; now, they have prices on their heads. In Paris this week, mathematicians unveiled a most-wanted list of seven of the most intractable math problems in the world. With a purse of $1 million for each, it is by far the biggest math prize ever announced and a call to action for mathematicians everywhere.

The announcement hearkens back to a similar exhortation, made in Paris a century ago. In 1900, at the second International Congress of Mathematicians, the German mathematician David Hilbert challenged his colleagues with 23 unsolved problems. "Hilbert's lecture, more than any other event, shaped 20th century mathematics," says Arthur Jaffe, former president of the American Mathematical Society and president of the Clay Mathematics Institute (CMI) of Cambridge, Massachusetts, which is sponsoring the challenge.[*] Unlike Hilbert, however, CMI is sweetening the deal with cash. Founded 2 years ago by mutual-fund magnate Landon Clay, the institute has a substantial endowment (Jaffe wouldn't name a figure) devoted to advancing mathematical knowledge. "If next year every one of these problems were solved, it wouldn't be a problem," Jaffe says. "It *would* be a surprise."

ILLUSTRATION BY JOE SUTUFF

P = NP? First on the list is a problem that could make computer encryption a thing of the past. The so-called "P versus NP" problem arose when computer scientists tried to figure out how efficiently algorithms crunch numbers. In general, the more data you cram into a computer program, the longer the program takes to process it. Consider an algorithm for alphabetizing a list of files. If you double the number of files, the program might take four times as long, on average, to put them in order. In the language of computer science, it is an n^2 algorithm. For most purposes, programmers are happy to come up with such "polynomial-time," or P, algorithms, as they aren't outrageously time-consuming to solve.

Even problems that don't seem to be solvable in polynomial time, such as factoring a large number, may be *checked* in polynomial time. To check whether someone has factored a large number, for instance, all you have to do is multiply the factors together. A problem checkable in polynomial time is called "NP." Clearly, all P algorithms are NP: If you can solve something in polynomial time, you can certainly check someone else's solution in polynomial time. In 1971, however, computer scientist Stephen Cook asked whether an NP algorithm is necessarily a P algorithm.

The answer appears to be "no"; NP problems such as factoring large numbers don't have any known polynomial-time solutions. But proving this is another matter—and a matter of some gravity. Mathematicians have proven that the hardest type of NP problems, called NP-complete, are equivalent. Thus a polynomial-time algorithm of one NP-complete problem can be tweaked to crack them all—including computer ciphers. "You could use [one] to break any encryption scheme," Cook says.

The Poincaré conjecture. In 1904, French mathematician Henri Poincaré was studying the classification of shapes in space, a field known as topology. One powerful way of classifying these shapes is to observe the behavior of shrinking loops of string on the surface of an object. For example, if you place a loop on the surface of a basketball, as it shrinks, it will always shrivel up into a point. On the other hand, a loop of string around a doughnut might not be able to shrivel completely: It could get stuck if it is looped around or through the doughnut's hole.

For two-dimensional surfaces such as the skin of a basketball or the glaze on a doughnut, the behavior of the shrinking loops completely describes the type of surface you're dealing with. If you know that every loop on a given surface shrinks to a point, then the surface is topologically equivalent to a sphere. Poincaré conjectured that the loop-closing test also holds true in the next dimension up, for three-dimensional surfaces. But he never proved or disproved his conjecture, and neither has any other mathematician.

The conjecture has been proven for every other dimension; the three-dimensional case alone remains. "The fact that the problem is still unsolved after so long is rather shocking," says John Milnor, a mathematician at the State University of New York, Stony Brook.

The Birch-Swinnerton–Dyer conjecture. This problem lies in the same realm of mathematics that Andrew Wiles used to prove Fermat's Last Theorem half a decade ago. Both hinge on the mathematical properties of geometric figures called elliptic curves: the set of points that solve an equation of the form $y^2 = x^3 + ax + b$. The Birch-Swinnerton–Dyer conjecture, which was formulated in the 1960s, is concerned with "rational" points on the curve, that is, points on the graph where both x and y are rational numbers. Associated with each such elliptical curve is a mathematical object called an "L-function"—basically, a formula that encodes the information about the curve in a different form. The conjecture states that there are an infinite number of rational points on a curve if and only if the curve's L-function equals zero at a certain value. Although the problem is abstract, it is related to questions about the areas of right triangles with rational-sized sides, which Cambridge mathematician John Coates calls the "oldest unsolved major problem in mathematics."

The Hodge conjecture. Like the Birch-Swinnerton–Dyer conjecture, the Hodge conjecture tries to link two mathematical concepts. In the branch of mathematics

[*] CMI's Web site is www.claymath.org. To be eligible for a prize, an entry must have been published in a peer-reviewed journal.

Turn-of-the-Century 'Hit List' Showed The Limits of Mathematical Ambition

David Hilbert wouldn't be thrilled by the fate of the 23 problems he challenged his colleagues to solve a century ago. Although his speech in 1900 helped set the course of mathematics in the 20th century, perhaps the most striking and depressing discovery in 20th century mathematics was that Hilbert's grand scheme was a fool's errand.

The Hilbert problems became the most-wanted list in mathematics. Some proved relatively easy. Hilbert's third problem, which dealt with cutting up tetrahedra, was solved within 2 years by Hilbert's graduate student Max Dehn. Others—such as number eight, the Riemann hypothesis—remain unsolved unto this day.

Even though the 23 problems span a wide variety of mathematical "flavors," many of them share an underlying theme. Hilbert desperately wanted to bring mathematics back to its rigorous, axiomatic roots, sweeping aside ad hoc assumptions and starting anew with a bare minimum of statements assumed to be true. Toward that end, Hilbert's second problem asked whether the axioms of logic can be proved to be consistent, while his fourth problem invited mathematicians to explore geometries similar to Euclidean geometry but with some of Euclid's axioms weakened or removed entirely. Hilbert's sixth problem asked whether physics can be axiomatized as mathematics had been. Ultimately, Hilbert wanted only a few axioms to govern logic, arithmetic, algebra, geometry, and all other areas of mathematical and scientific thought.

At the turn of the century, this minimalist vision seemed like a reasonable, if ambitious, program. A new branch of mathematics, set theory, held hints of a power deep enough to unify all mathematical thought. By invoking only a few axioms to lay down the laws of manipulating sets of objects or similar concepts, mathematicians could set the foundations of logic, create all the numbers, build up the rules of arithmetic, and then proceed onward to geometry and other pursuits. This idea reached its culmination in the 1920s, when Bertrand Russell and Alfred North Whitehead started with a mere handful of axioms and then took roughly 1000 pages of dense mathematical scribbling to prove that 1 + 1 = 2.

Hit man. David Hilbert's grand program fizzled, but his 23 problems transformed mathematics.

But the Hilbert program was doomed by yet another mathematician, Kurt Gödel. In 1931, Gödel proved that no self-consistent set of axioms is sufficient to prove everything that is true—that there are always true theorems beyond the ken of whatever axioms you choose.

Gödel's incompleteness theorem swept away the Hilbert program. There is no overarching set of axioms that allows you to codify the whole of mathematics. Gödel's melancholy conclusion was perhaps the most significant discovery in 20th century mathematics.

Unlike Hilbert's problems, the seven problems chosen by the Clay Mathematics Institute (CMI) in Cambridge, Massachusetts, do not share a unifying vision. Nor do they hew to the cutting edge of mathematics. "It's exactly the opposite," says Arthur Jaffe, president of CMI. Instead of trying to promote a mathematical "program" as Hilbert did, Jaffe says the CMI chose to "focus on classical problems, each very important, each having resisted solution." But mathematicians will be grateful to CMI if its awards affect mathematics a fraction as much as did Hilbert's failure.

–C.S.

known as algebraic geometry, mathematicians try to combine abstract algebra, which studies the relations and symmetries of numbers, with geometry, which studies shapes in various spaces. Hodge cycles are structures that have a great deal of algebraic power but no obvious geometric interpretation. Algebraic cycles have a geometric interpretation—they are related to the intersection of curves in space—but are less powerful algebraically. The Hodge conjecture links the two, stating that a Hodge cycle can be written as a sum of algebraic cycles, combining the power of the former and the easy interpretation of the latter.

Yang-Mills existence and mass gap. Problem number 5 is inspired by a branch of physics known as Yang-Mills theory, which describes particles by using the language of mathematical symmetries. Even though Yang-Mills theory has enabled physicists to unify the electromagnetic, weak, and strong forces, it's not certain that reasonable solutions to Yang-Mills equations actually exist—and if they do, whether those solutions will have a "mass gap" that explains why physicists can't isolate quarks. "There's no real outline or idea [for] how to go about this," Jaffe says.

Navier-Stokes existence and smoothness. This problem concerns a set of differential equations that describes the motion of incompressible fluids: the Navier-Stokes equations. Although they're relatively simple-looking, the three-dimensional Navier-Stokes equations misbehave badly. "You can set up Navier-Stokes with nice, smooth, reasonably harmless initial conditions," and the solutions can wind up being extremely unstable, says Princeton mathematician Charles Fefferman. "People think they see breakdowns—a singularity develops. It appears to be very, very bad." If mathematicians could tame the outrageous behavior of Navier-Stokes, it would dramatically alter the field of fluid mechanics. "To understand the behavior of fluids would have a very big effect in science and technology, and also in mathematics," Fefferman says.

The Riemann hypothesis. No most-wanted list would be complete without this, the granddaddy of mathematical mysteries. The hypothesis was first published in 1859 by German mathematician Bernhard Riemann, who was investigating the properties of the so-called zeta function: $\zeta(s) = 1 + 1/2^s + 1/3^s + 1/4^s + \ldots$. No matter what positive number you plug in for s, you never get $\zeta(s)$ to equal zero. However, this is not true in the realm of complex numbers—numbers that can be expressed as $a + bi$ where i is the square root of -1. In fact, infinitely many "zeros" of the zeta function contain a multiple of i, and they all seem to have a real part of 1/2; that is, they equal $1/2 + bi$ for some real number b. "Seem" is the key word, however. Although more than a billion known zeros follow the pattern, no one has proved that they all do.

If the hypothesis is true, it affects almost all other branches of mathematics—for instance, it will tell mathematicians about the distribution of prime numbers. "To me it is the central problem in pure mathematics, much more so today than it was 50 years ago," says Enrico Bombieri, a mathematician at the Institute for Advanced Study in Princeton, New Jersey. The zeta function is closely related to the L-functions of algebraic geometry, for instance, so the Riemann Hypothesis affects the same areas of mathematics as did Wiles's proof of Fermat's Last Theorem. "The connection with other parts of mathematics is getting deeper," Bombieri says.

–CHARLES SEIFE