BOOKS: MATHEMATICS

Connections Through a Matrix

Herbert S. Wilf

Proofs and Confirmations is one of the most brilliant examples of mathematical exposition that I have encountered, in many years of reading the same.

In the early 1980s, William Mills, Howard Rumsey, and David Robbins put forth the alternating sign matrix (ASM) conjecture. It offered a formula for the number of alternating sign matrices (objects that generalize permutation matrices) of size $n \times n$. These are square matrices of 1s, −1s, and 0s for which (i) the sum of the entries in each row and in each column is 1 and (ii) the nonzero entries of each row and of each column alternate in sign. One such matrix is

Proofs and Confirmations
The Story of the Alternating Sign Matrix Conjecture
by David M. Bressoud

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$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

By playing around with a computer, you could easily convince yourself that there are 429 different 5×5 ASMs; 7436 different 6×6 ASMs; 218,348 7×7 different ASMs; and so forth. You might then wish for a general formula that tells how many $n \times n$ ASMs exist. Mills, Robbins, and Rumsey found a formula that fit all of the data their computer had produced: the number of different $n \times n$ ASMs is exactly

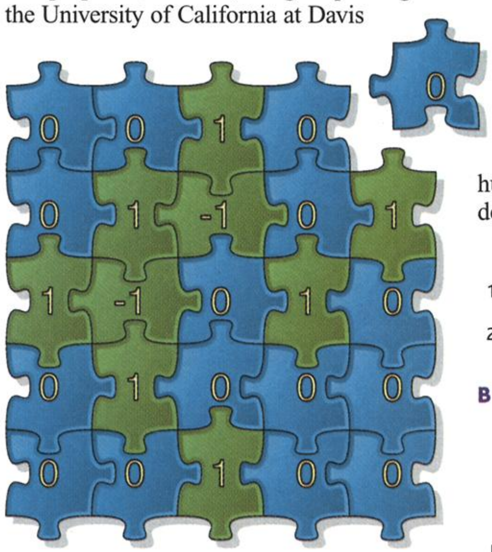
$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

So, for example, the number of 5×5 ASMs is

$$\frac{1!}{5!}\frac{4!}{6!}\frac{7!}{7!}\frac{10!}{8!}\frac{13!}{9!}$$

which does work out to be 429. But they did not prove that their formula is valid for all n; they only conjectured that it was.

The problem remained tantalizingly unsolved until the mid-1990s, when Doron Zeilberger of Temple University proved the conjecture completely. His 1995 proof (*1*) is about 84 pages long. Scores of people worldwide refereed the paper; each was responsible for one node of an organizational tree of the paper that Zeilberger had prepared. In 1996, Greg Kuperberg of the University of California at Davis found another proof of the ASM conjecture. His proof (*2*) uses results from the theory of "square ice" (a two-dimensional arrangement of water molecules) developed by physicists working in statistical mechanics.



Explaining the solution. The numbers in this picture puzzle form an alternating sign matrix. And there are 429 ways to arrange pieces with the green or blue shapes in a 5 x 5 tray that has notches on the sides and knobs on the top and bottom.

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The search for the proofs involved material drawn from a wide range of mathematical topics including the theory of partitions, lattice paths, plane partitions, symmetric functions, the Macdonald conjecture, and an algorithm for evaluating determinants discovered by Charles Dodgson (better known as Lewis Carroll). Bressoud has tackled the unenviable task of explaining this entire cluster of ideas to the widest possible audience. He has succeeded. The book could form the basis for an advanced undergraduate or graduate course. For students who have previously learned the basics of combinatorial mathematics, the course could be more or less self-contained. For those to whom the notion of, say, generating functions, is foreign, the instructor would need to intersperse such background material at several points.

The author provides very careful and stimulating explanations of the many connected topics and of the context in which the problem resides. He accompanies his discussions with a large and rich collection of exercises of varying difficulty. Thus, with study, one can gain the broad picture of the many threads that were pulled together by Zeilberger and by Kuperberg in their respective proofs, together with all of the devilish details.

This is not for the faint-hearted, nor is *Proofs and Confirmations* a book that can be read in an easy chair, like a novel; it demands active participation by the reader. But Bressoud rewards such readers with a panorama of combinatorics today and with renewed awe at the human ability to penetrate the deeply hidden mysteries of pure mathematics.

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References and Notes

1. D. Zeilberger, *Electron. J. Combinatorics* **3**(2), R13 (1996). Available at www.combinatorics.org
2. G. Kuperberg, *Int. Math. Res. Notes* **1996**, 139 (1996).

BOOKS: ZOOLOGY

Fishing Stories

Axel Meyer

Just before Christmas 1938, Majorie Courtney-Latimer, the curator at a small museum in East London, South Africa, found a strange-looking fish at the local docks. She sent a brief description and rough sketch of it to the South African ichthyologist J. L. B. Smith. The bony head plates, heavy scales, and limb-like fins were sufficient for Smith to identify the new species, which, in honor of its discoverer, he named *Latimera chalumnae*, as a living coelacanth. The fish belonged to a group that was thought to have gone the way of the dinosaurs 70 million years ago. The oldest-known coelacanths appeared in the Devonian (before 375 million years ago), and some paleontologists saw the group as ancestral to the earliest terrestrial vertebrates. Study of this "living fossil" could facilitate our understanding of the vertebrate's

A Fish Caught in Time
The Search for the Coelacanth
by Samantha Weinberg

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