MATHEMATICS

Fermat's Last Theorem's First Cousin

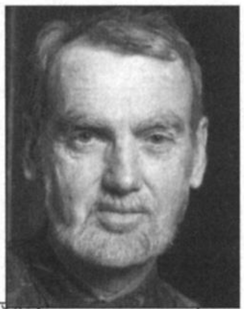
A proof of the Langlands conjecture for function fields answers a question that has puzzled mathematicians for over 3 decades

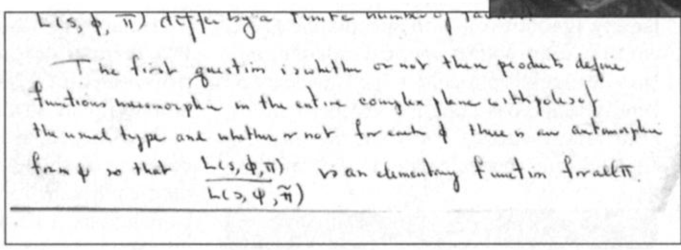
In January 1967, a 30-year-old Princeton mathematics professor named Robert Langlands wrote to André Weil, the dean of the world's number theorists, asking for his opinion about two new conjectures. "If you are willing to read [my letter] as pure speculation I would appreciate that," wrote Langlands; "if not—I'm sure you have a waste basket."

Weil never wrote back, but Langlands's letter turned out to be a Rosetta stone linking two different branches of mathematics. He posited that there was an equivalence—rather like a French-English dictionary—between Galois representations and automorphic forms. The former describe the intricate relationships among the solutions to equations studied in number theory. The latter are highly symmetric functions. The most familiar examples are the sine and cosine functions, which are periodic, or invariant under horizontal shifts. Such shifts (for example, "move left 2π units" or "move right 4π units") give the same result when performed in any order. The elementary symmetry of the sine and cosine functions is as boring to mathematicians as a test pattern. But Langlands foresaw that the future of number theory lay in understanding functions with more exotic, order-sensitive kinds of periodicity—functions with the infinite complexity of fractals.

For 30 years, Langlands's questions—which are often called the "Langlands program" because of their many ramifications—have been a driving force in number theory. The program has led to two Fields medals (widely considered the equivalent of the Nobel Prize for mathematics) for other mathematicians. Perhaps the greatest mathematical achievement of the 20th century, Andrew Wiles's 1994 proof of Fermat's Last Theorem, can also be viewed as the completion of a small part of the Langlands program. "The Langlands program ties together theories that are a priori very different and very distant from one another," says Laurent Lafforgue, a number theorist at the Université de Paris-Sud.

This fall, thanks to Lafforgue, another piece of the program finally fell into place. In November, Lafforgue gave the first U.S. presentation of his proof of the "Langlands conjecture for function fields" in a series of lectures at the Institute for Advanced Study in Princeton, New Jersey. A 300-page handwritten version of Lafforgue's proof has been circulating among mathematicians since the summer, but it has not yet been submitted for publication. Nevertheless, the experts seem quite confident that it will hold up. "I'm sure it's a contender for the Fields medal," says Peter Sarnak, a number theorist at Princeton University. Says Langlands, who attended the lectures, "There's nothing suspicious about his argument. He was a man who knew exactly what he was talking about."




$L(s, \phi, \pi)$ differ by a finite number of

The first question is whether or not these products define functions meromorphic in the entire complex plane with poles of the usual type and whether or not for each ϕ there is an automorphic form ψ so that $\frac{L(s,\phi,\pi)}{L(s,\psi,\tilde{\pi})}$ is an elementary function for all π.

Number theorist. Robert Langlands made his eponymous conjectures in a 1967 letter to André Weil.

The "fields" that Lafforgue refers to have nothing to do with fields in physics (or, for that matter, agriculture). Mathematicians use the word "field" to denote any algebraic structure consisting of objects, or elements, that can be added, subtracted, multiplied, and divided according to the rules that govern real numbers. In one large category of fields, number fields, the objects are ordinary rational numbers (fractions), or real numbers, or complex numbers. Many number theorists deal only with these fields, where the classical questions of number theory (such as Fermat's Last Theorem) first arose.

However, there are two other types of fields, each distinctive enough in its properties to attract its own coterie of experts. Perhaps the most bizarre are local fields such as the 7-adic numbers, where live such creatures as the number $1 + 7 + 7^2 + 7^3 + \ldots$ These numbers are forbidden in the universe of real numbers because they can become infinitely large. Finally, there are the function fields, whose elements are polynomials or quotients of polynomials—for example, $(x^2 - 3x + 1)/(x + 2)$. Although local fields and function fields are less familiar than number fields, they often are easier to study. Lafforgue's work proves Langlands's conjectures only in the context of function fields. In 1998, three other mathematicians proved them for local fields as well. That leaves only the central problem of number fields unresolved. "I don't think it will take a miracle," says Langlands. "There's a hump we've got to get over, an insight that's not out there yet."

Historically, Langlands's conjectures arose out of an effort to find very general versions of what number theorists call reciprocity laws—patterns governing how whole numbers can be broken down into sums of products of other whole numbers. (The term is a bit of a misnomer, as reciprocity laws have nothing particularly to do with reciprocals.) These laws date back hundreds of years. In the 17th century, Pierre de Fermat enjoyed solving such questions as this: Which prime numbers can be represented as a sum of two squares? For example, 5 is $2^2 + 1^2$; but 7 cannot be written as a sum of squares. He discovered a simple pattern, whose reasons were nevertheless mysterious: An odd prime number can be written as a sum of two squares if it is 1 greater than a multiple of 4, and not if it is 3 greater than a multiple of 4. Thus the pattern is periodic in the same sense that the sine and cosine functions are. As far as representations as sums of squares are concerned, a shift of the whole prime number system by four units to the left would be invisible.

Over the centuries, mathematicians discovered a host of other reciprocity laws. The story seemed to reach a glorious conclusion in 1927, when Emil Artin proved a single reciprocity law that encompassed all the others. Although the number theory behind his work was profound, the geometry was rather banal—the "symmetry groups," or sets of periodicities involved, were one-dimensional, like the periodicities of the sine and cosine functions. More complicated patterns eluded mathematicians until 1967, when Langlands's link to automorphic forms showed mathematicians how they could bring to bear the theory of n-dimensional matrices (or n-by-n tables of

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numbers). A number, Langlands realized, is just a 1-by-1 matrix in disguise. Just as shifts can be represented by a single number or 1-by-1 matrix—the distance shifted—he hypothesized that the transformations behind more general reciprocity laws could be represented by matrices. The link has remained conjectural, but the confirmation of one special case led to disproportionately large consequences. The seed of Andrew Wiles's monumental work on Fermat's Last Theorem was his proof of Langlands's conjecture for 2-by-2 matrices whose entries are all 0, 1, or 2.

Lafforgue's tour de force is unlikely to have such dramatic consequences, because function fields lack the éclat of number fields and Fermat's Last Theorem. However, other mathematicians say that its significance will likely become apparent in time. "Lafforgue has proved that two very different-looking things are the same," says Nicholas Katz, an algebraic geometer at Princeton University. "When you do that, it's almost always the case that there are some properties that are very easy to see one way, and incredibly obscure the other way. … It's too soon to predict exactly how it's going to work, but I feel strongly that it's very important."

–DANA MACKENZIE

ECOLOGY

How Climate Change Alters Rhythms of the Wild

The more scientists look, the more connections they see between shifts in climate and changes in animal behavior and populations

Each year for 31 years, biologist Jerram Brown has trekked into the Chiricahua Mountains of southern Arizona to chronicle the rites of spring for a population of Mexican jays. Brown adheres to his own ritual: The biologist from the State University of New York, Albany, notes on which date the females lay their first clutch, then several weeks later he shinnies up 15-meter-tall Chihuahua pines to band each and every chick. His perseverance has paid off with an intriguing observation: The jays are laying their eggs earlier and earlier each season. By 1998, the first eggs of the season arrived 10 days earlier than in 1971.

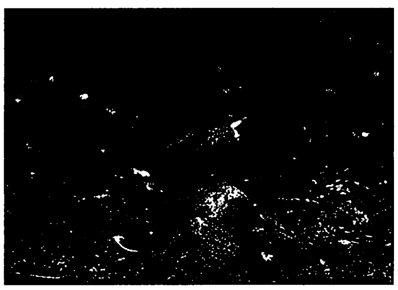
Early bird. Warmer nights seem to be why Mexican jays become parents sooner.

Brown blames global warming for turning the hands forward on the jays' reproductive clock. Although Arizona hasn't necessarily gotten hotter, it has grown less cool. In the months leading up to the breeding season, Brown found, average daily minimum temperatures have nudged up 2.7 degrees Celsius in 27 years. A narrower temperature range probably encourages earlier breeding by allowing birds to conserve energy on cold nights, when they can burn off about 10% of their weight just to stay warm, Brown says. The warmer air may also roust insects earlier, which would likewise provide extra calories for females to funnel into the energetic business of egg production. Strengthening the case against warming, Brown says, is the fact that many other species in the Northern Hemisphere, from birds to frogs, are also breeding earlier than they were years ago. "While no one study can prove that earlier breeding is caused by global warming," he says, "it all fits in."

For more than 2 decades, climate modelers have warned that global warming may transform our environment by pushing corn belts north, expanding deserts, and melting ice caps. Now biologists are getting in on the action, compiling an impressive array of data suggesting that climate changes big and small can have profound effects on species. Climate's fingerprints are turning up in observations compiled over years and decades.

The sheer complexity of ecosystems makes biologists reluctant to start predicting the fate of individual species based on various climate-change scenarios. But with some models forecasting that average global temperatures could rise as much as 4.6 degrees Celsius in the coming century, the new observations "give us a handle to think about where things will be in 2100," says biologist Camille Parmesan of the University of Texas, Austin.

Dippers take a dive. The latest observation of a species' response to climate change involves Norway's national bird, the dipper. On page 854 of this issue, a team led by Bernt-Erik Sæther and Jarle Tufto of the Norwegian University of Science and Technology in Trondheim reports that while the number of dippers in a population in southern Norway fluctuated between 1978 and 1997, it followed an upward trajectory. The trend—based on an analysis of more than 20 years of observations by amateur bird watchers—was closely associated with the gentle touch of an atmospheric pressure system called the North Atlantic Oscillation (NAO) and the milder winter temperatures it brought.



Warm-weather dipper. A gentle NAO boosts populations of this bird.

The NAO, which embraces much of the Northern Hemisphere, delivers warm, wet winters to northern Europe during its high phase. (When it flips to low gear, bitter cold usually sets in.) For much of the past 3 decades, the high phase has dominated, and for the dipper, "a warm year is a good year," Sæther says. Because the bird dives for its food on stream bottoms, it has little to eat if streams ice over. The researchers found that the bird's ranks swelled after warm years, thanks to increased immigration and a higher birth rate in the local population. A long-term warming of 2.5 degrees Celsius, Sæther's team estimates, should boost dipper numbers by 58%.

The team's mathematical model, which takes into account random population fluctuations and temperature changes, can be applied to other species as well, says Peter Kareiva, a biologist with the U.S. National Oceanic and Atmospheric Administration in Seattle. Unlike previous models, Sæther's teases apart the effects of climate change from those of density dependence, a phenomenon in which mortality rates tend to rise, and birth rates fall, as a population's size increases.

Also in tune with the NAO, it would appear, are grazing mammals. Eric Post and Nils Chr. Stenseth of the University of Oslo in Norway analyzed 15 years of data on northern mammals. They found that 9 of 11 ungulate populations—including caribou, musk ox, moose, feral goats, and Soay sheep—declined following warm NAO winters. But the effects varied by location. In maritime areas, the survival rates of Soay sheep and feral goats improved during mild winters,

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