BOOKS: MATHEMATICS

True Stories of An Imaginary Number

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The equation $e^{i\pi} + 1 = 0$ is one of the most important in mathematics, linking five of the most significant quantities in our number system. Stories about 0 and 1 have been around for years, and two recent books have provided "biographies" of e and π (*1*). *An Imaginary Tale* completes the series, giving us the story of i (or $\sqrt{-1}$, to use its older name). Like its predecessors, this volume discusses not only the history of its topical quantity, but also many of that number's uses, both in earlier times and today. And the author tells this story with much grace, wit, and charm.

An Imaginary Tale
The Story of $\sqrt{-1}$
by Paul J. Nahin
Princeton University Press, Princeton, NJ, 1998. 277 pp. $24.95, £18.95. ISBN 0-691-02795-1.

Paul J. Nahin, an electrical engineer from New Hampshire, has provided an outstanding book, which integrates the history of imaginary numbers with a lot of interesting mathematics. Although the author claims that large chunks of his book can be understood by high school seniors, there is much in it that will be new to college mathematics majors or to professional scientists and engineers. To give an example from the first chapter: It is certainly a familiar fact that we can find the real roots of a polynomial equation by checking where the graph of the corresponding function crosses the x-axis. Most students are also aware that if a polynomial of degree n crosses the x-axis in r places, there are $n - r$ complex roots, which come in conjugate pairs. Although these pairs cannot be seen on the graph, Nahin shows us that there is a simple way, using a ruler, to read the complex roots from the graph.

Nahin begins his book with the wonderful stories of the solution of the cubic equation by Scipione del Ferro, Niccolo Tartaglia, and Girolamo Cardano, and of Rafael Bombelli's discovery of complex numbers in the context of the so-called irreducible case of the cubic. We then learn of Francoise Viète's method for solving such equations via trigonometry, of René Descartes' geometric methods for solving polynomial equations, and of John Wallis' attempts to represent complex numbers in the plane. Our current geometric treatment of complex numbers comes from the work of Caspar Wessel, Jean-Robert Argand, and Carl Friedrich Gauss, all of whose contributions are discussed in detail. Nahin also explains the more abstract treatment, in the work of William Rowan Hamilton, of complex numbers as pairs of real numbers.

The book's next several chapters exemplify the remark of Jacques Hadamard that "the shortest path between two truths in the real domain passes through the complex domain." We see how complex numbers are, in fact, used to solve very real problems in mathematics and, reflecting Nahin's profession, especially in physics and engineering. Kepler's laws of planetary motion easily fall out of a consideration of complex exponentials, as do important properties of electrical circuits. Also among the various applications that Nahin discusses so effectively are Euler's developments of the zeta and gamma functions.

Bombelli's book. This comprehensive account included contributions on the algebra of complex numbers.

Many of the results in the chapters on uses of complex numbers are surprising (and perhaps awe-inspiring), but it is the final chapter, on Augustin-Louis Cauchy and complex function theory, which will probably prove most valuable to anyone studying complex analysis. For although Nahin repeatedly claims that he is not writing a textbook, this chapter provides a far better account of the beginnings of that theory than do most texts on the subject: The origins and meaning of the Cauchy-Riemann equation are succinctly discussed. Several examples of the use of Cauchy's integral theorem are expertly explained. And Cauchy's development of the techniques of contour integration in calculating real integrals is masterfully illustrated with important examples, including one giving a derivation of Kepler's third law.

The one possible criticism of this chapter, and indeed of the entire book, is that there are a small number of historical errors. For example, Nahin claims that Green's theorem relating a line integral around a closed curve in the plane to a double integral over the curve's interior (a theorem used in the proof of Cauchy's integral theorem) appeared in George Green's famous essay of 1828. There are theorems in the essay from which one could derive Green's theorem, but Green himself never stated or used that result. Nahin compounds his error by accusing William Thomson of mentioning the theorem without attribution in a letter to his friend George Stokes. The theorem Thomson wrote about was, however, what is today known as Stokes' theorem, which relates a line integral around a closed surface in three dimensions to the surface integral over that surface. And it was not Green's theorem that Mikhail Ostrogradsky first stated and proved, but the divergence theorem (relating a surface integral to a volume integral). Although all of these theorems are special cases of a more general result, that fact was not discovered until the end of the 19th century.

Among other minor errors are a mistranslation of the title of Copernicus' masterwork, *On the Revolutions of the Heavenly Spheres*, which leads to some confusion about Copernicus' intention; the notion that Hamilton associated time with $\sqrt{-1}$ in the essay in which he developed the notion of complex numbers as ordered pairs of real numbers, when in fact time had to do with Hamilton's conception of positive and negative quantities; and the statement that Newton's reason for not using calculus in his *Principia* was to avoid a philosophical debate on the merits of the calculus. Actually, Newton used geometry because he felt his readers would be most familiar with arguments in that style; even so, many of his arguments utilized infinitesimals, quantities to which Euclid's axioms do not apply.

These minor errors not withstanding, *An Imaginary Tale* is marvelous reading and hard to put down. Readers will find that Nahin has cleared up many of the mysteries surrounding the use of complex numbers. And should anyone challenge you to show that these numbers are just as "real" as the real numbers, you will find no better answer than to quote the first 30 decimal places of i^i: 0.207879576350761908546955619834.

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References

1. E. Maor, *e: The Story of a Number* (Princeton Univ. Press, Princeton, NJ, 1994); P. Beckmann, *A History of π* (Golem, Boulder, CO, 1970).