PERSPECTIVES: GENERAL RELATIVITY

A New Twist

Matt Visser

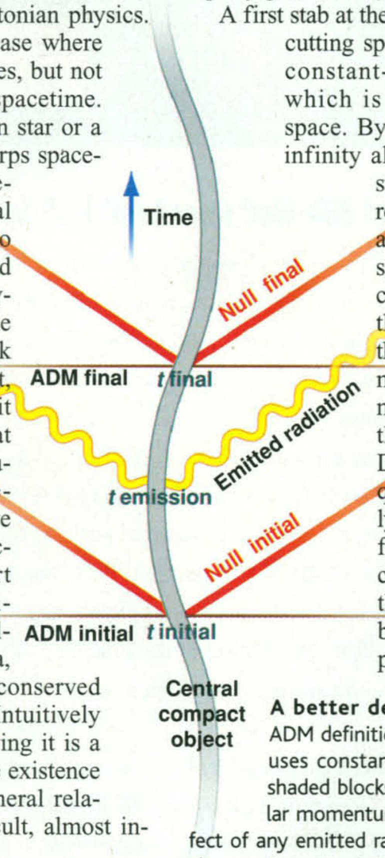
Many of the standard concepts of Newtonian physics, such as energy, momentum, and angular momentum (the momentum carried by rotating objects), are surprisingly elusive in Einstein's general relativity. The problem is that the geometry of space and time is itself dynamic, so we do not know exactly what the static foundations should be. Archimedes said "give me a place to stand and I will move the Earth." It is the lack of a definite place to stand that makes definitions of mass and momentum so tricky in general relativity.

Enhanced online at www.sciencemag.org/content/full/282/5387/249

If gravitational fields are weak everywhere in the universe, so that spacetime is more or less flat and things don't move too fast, you get standard Newtonian physics. Much more vexing is the case where the gravity is strong in places, but not strong everywhere in the spacetime. For an object (like a neutron star or a black hole) that strongly warps spacetime, deep down in the region of strong gravitational field it might be difficult to define mass, momentum, and angular momentum. However, far enough away from the object, where gravity is weak and spacetime is nearly flat, physicists will assert that it is "intuitively obvious" that mass, momentum, and angular momentum can be defined without having to delve deeply into its interior structure. They will also assert that it is "intuitively obvious" that these asymptotically defined masses, momenta, and angular momenta are conserved as in Newtonian physics. Intuitively obvious it may be, but proving it is a different matter: Proving the existence of conservation laws in general relativity is an extremely difficult, almost insanely difficult, proposition.

In a recent report (*1*), Anthony Rizzi of Princeton University has gone a long way toward doing this: He has provided an improved, mathematically precise definition of angular momentum in general relativity with all the required properties. To even begin the construction, one must consider asymptotically flat spacetimes: Toward infinity, either along a constant-time slice (spatial infinity) or outward at the speed of light (null infinity), the spacetime geometry has to become flatter and flatter, and in the limit it should be exactly flat. Mass and momentum are then constructed as components of a four-dimensional vector in this flat spacetime at spatial or null infinity. Angular momentum is a little trickier but can also be thought of as a vector in this flat spacetime. (There are a number of rather horrifying technical incantations that I am sweeping under the rug even to get this far, and the situation rapidly gets worse.)

A first stab at the problem is provided by cutting spacetime into a stack of constant-time slices, each of which is a three-dimensional space. By moving out to spatial infinity along one of these constant-time slices and surrounding the object with a big, more-or-less-spherical surface, one can use derivatives of the metric, plus Gauss' theorem, to define mass, momentum, and angular momentum. These are the so-called Arnowitt-Deser-Misner (ADM) quantities. Unfortunately, these definitions are for some purposes too crude: The properties of the object itself cannot be separated from the properties of any radiation it might be emitting (see figure). The limit used to define ADM mass, momentum, and angular momentum involves mathematically moving outward along a constant-time slice, effectively at infinite speed, thereby overtaking any outgoing radiation and lumping it in with the total mass, momentum, or angular momentum.



A better definition. The standard ADM definition of angular momentum uses constant-time slices (horizontal shaded blocks). Hence, the ADM angular momentum always includes the effect of any emitted radiation, and so it "never" changes. In contrast, Rizzi's new definition of angular momentum (*1*) uses null surfaces (red lines) that sweep outward at the speed of light. Two different null surfaces can trap outgoing radiation between them, and the angular momentum can thus be defined before and after emission. The change in angular momentum during emission is equal to the angular momentum carried off by the radiation.

Improved definitions of mass and momentum account for emitted radiation by taking a limit along a surface that itself moves out at the speed of light. This null limit does not overtake any outgoing radiation and lets you isolate the properties of the central object itself, without being contaminated by the radiation that has already escaped. For mass and linear momentum, the situation has been pretty much under control since the 1960s: The Bondi mass and momentum (*2*) are defined in terms of such a limit along surfaces sweeping outward at the speed of light. The difference between the ADM and Bondi masses is a measure of how much radiation has already been emitted.

To extend this procedure to angular momentum is not easy and has taken an extra 30 years (*1*). The basic idea is that a rotating object generates a gravitational field with a "twist." This effect is measurable because it affects the geodesics and so will influence gravity wave detectors. Even far away from the source, there is a characteristic "imprint at infinity" that rotating sources impart to the spacetime. This imprint, this twist in the pattern of geodesics, is used to define angular momentum.

The new definition given in (*1*) has many desirable properties: It correctly incorporates the notion of twist, it provides a quantity that is in principle measurable, it gives the correct answers for test cases like Schwarzschild and Kerr black holes, it is unique (up to choice of origin and rest frame), it is conserved in a suitable sense, and the rate of change of the angular momentum can be related to the angular momentum carried off by gravitational radiation.

Every time a new advance has been made in basic definitions, a minor industry has bloomed as theorists apply the definition to every conceivable spacetime, sometimes even to spacetimes of astrophysical interest. After the definitions of Bondi mass and Bondi momentum were developed, it led to great interest in "relativistic rocket" spacetimes, objects that acquire net momentum by anisotropic emission of gravitational radiation. With this new definition in hand, it is now possible to attack problems of spin-up and spin-down in a consistent and rigorous manner.

References

1. A. Rizzi, *Phys. Rev. Lett.* **81**, 1150 (1998).
2. H. Bondi, M. G. H. Van de Burg, A. W. K. Metzner, *Proc. R. Soc. London A* **269**, 21 (1962).

The author is in the Physics Department, Washington University, St. Louis, MO 63130–4899, USA. E-mail: visser@kiwi.wustl.edu