from the crowd of 3500 mathematicians assembled in Berlin's International Congress Center came when the IMU presented the conqueror of Fermat's Last Theorem with a special one-time tribute. In 1994, the last time the Fields Medals were awarded, a gap remained in Wiles's proof. With the gap long since repaired, many believe that Wiles—now 45—has produced results of such rare beauty and significance that the IMU should have made an exception to its "no older than 40" rule and awarded him a Fields Medal. One mathematician remarked that Wiles has already gotten so many prizes he doesn't need a Fields Medal. No, said another, the Fields Medal needs Wiles.

–ALLYN JACKSON

Allyn Jackson is a writer and editor for the *Notices of the American Mathematical Society* and lives in Munich, Germany.

MATHEMATICS

Packing Challenge Mastered At Last

Johannes Kepler is best known for his elliptic laws of planetary motion. But mathematicians also remember him for a vexing problem in geometry: proposing—but not proving—that the densest possible packing of same-sized spheres is the arrangement familiar today to anyone who's ever admired a pyramid of oranges in a grocery store. Known to crystallographers as the face-centered cubic lattice packing, it fills a little over 74%—$\pi/\sqrt{18}$, to be precise—of space.

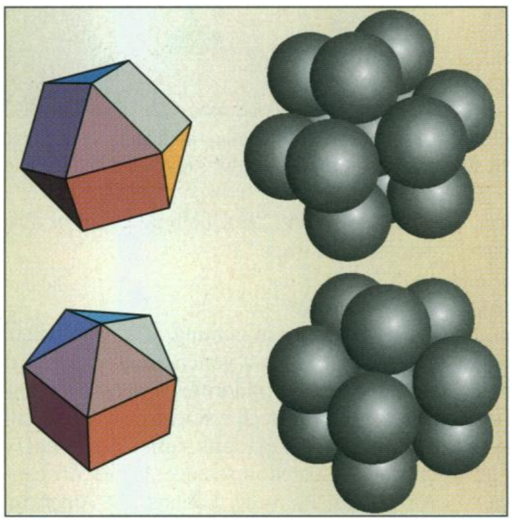
For nearly four centuries Kepler's conjecture has remained one of those mathematical Everests, like Fermat's Last Theorem, that people tackle for the sheer challenge of it. There's never been much doubt that the conjecture is true; the question has always been whether anyone can prove it. The answer, finally, appears to be Yes.

Thomas Hales, a mathematician at the University of Michigan, Ann Arbor, recently announced the completion of a lengthy analysis that appears to provide a rigorous proof of Kepler's conjecture. Hales's analysis, parts of which have already been published, combines 250 pages of mathematical reasoning with computer programs that enumerate and check thousands of crucial details. "It's really an amazing achievement," says Neil Sloane, a sphere-packing expert at AT&T Laboratories. Hales's proof has yet to undergo close scrutiny, but Sloane is impressed with what he's seen so far. "He's documented everything very carefully," Sloane says. "Nobody's raised a single doubt about his work."

That's important, in light of the Kepler conjecture's recent history. In 1990, Wu-Yi Hsiang, a mathematician at the University of California, Berkeley, announced that he had solved the sphere-packing problem (see *Science*, 1 March 1991, p. 1028). However, Hsiang's proof encountered a buzz saw of criticism from experts, including Hales, who said there were numerous flaws and gaps in the proof's reasoning (see *Science*, 12 February 1993, p. 895). In 1994, Hales wrote a detailed critique of Hsiang's proof for *The Mathematical Intelligencer*, calling on Hsiang to withdraw his claim. (Hsiang reportedly stands by the correctness of his proof, but could not be reached for comment.)

Hales's own proof follows a strategy first outlined by the Hungarian mathematician László Fejes Tóth in 1953. The strategy is to reformulate the conjecture in "local" terms, reducing it from a question about infinitely many spheres filling all of space to a series of questions about how certain finite arrangements of spheres fit together.

To carry out that strategy, Hales invented a new way to allocate the empty space in a packing to individual spheres. The standard approach assigns to each sphere its "Voronoi cell," consisting of the points that are closer to it than to any other sphere. If every sphere in any other packing simply occupied no more than 74% of its Voronoi cell, the Kepler conjecture would follow immediately. But Voronoi cells don't give a consistent measure of a packing's density. The spheres in some packings occupy as much as 75.5% of their Voronoi cells, although this high density is invariably canceled out by the low density of nearby cells.



S. FERGUSON

Packing jobs. Face-centered cubic (top) is the most efficient way to pack spheres, but a rival arrangement, the pentahedral prism, was hard to rule out.

Hales calls his alternative a "star decomposition." Roughly speaking, a star is a modified Voronoi cell with a batch of tetrahedral protuberances. A second key ingredient of his proof is a novel way to "score" the local density of each star. In Hales's convention, the stars in the face-centered cubic lattice packing all have a score of 8 points. Any counterexample to the Kepler conjecture would have to include stars with a score greater than 8.

"The proof gives a classification of all the stars that can potentially be a counterexample to the Kepler conjecture," Hales says. The list is a long one, but finite: A computer program found 5094 different types of stars, any one of which could conceivably have a score greater than 8. Each type then had to be ruled out, by showing that the largest score for each type of star stayed less than 8.

To do so, Hales had a computer convert each inequality estimate into a series of problems in linear programming, a mathematical method that is widely used for optimization problems in industry. So that no star with a winning score would slip through the gaps of round-off error, the computer did the conversions using a mathematically rigorous technique called interval arithmetic. Most of the cases were easily disposed of, but some required extra care. "At the very end, it came down to 50 or so cases that the general arguments didn't rule out," Hales recalls. Hales and a graduate student, Sam Ferguson, looked at those cases one by one.

One particularly problematic case, a local arrangement called a pentahedral prism, consisting of 12 spheres surrounding a 13th central sphere, became the subject of Ferguson's doctoral dissertation. "I only handled one case," Ferguson notes proudly. "It just ended up being the worst case." Initial calculations only said the pentahedral prism had a score less than 10. Ferguson had to refine the analysis to get a bound below 8.

Early this month, the final pieces fell into place: The last potential counterexamples had been eliminated. Hales is careful not to claim too much. "This is not a refereed paper, and so it should be taken accordingly," he says. "I think [the experts] should be able to judge the overall strategy and methods and that sort of thing quite quickly, but I expect it'll be several months before the details will have been carefully checked."

Assuming all goes well, proofs of other sphere-packing conjectures might follow. "I'll be curious to see what other problems can be solved by similar methods," Hales says. One possibility: A computationally intensive approach could suffice to prove the Kepler conjecture in four dimensions rather than three—a packing you'll never see at the grocer's.

–BARRY CIPRA