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Neptune's Eccentricity and the Nature of the Kuiper Belt

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The small eccentricity of Neptune may be a direct consequence of apsidal wave interaction with the trans-Neptune population of debris called the Kuiper belt. The Kuiper belt is subject to resonant perturbations from Neptune, so that the transport of angular momentum by density waves can result in orbital evolution of Neptune as well as changes in the structure of the Kuiper belt. In particular, for a belt eroded out to the vicinity of Neptune's 2:1 resonance at about 48 astronomical units, Neptune's eccentricity can damp to its current value over the age of the solar system if the belt contains slightly more than an earth mass of material out to about 75 astronomical units.

Beyond the orbit of Neptune lies a recently discovered disk of objects called the Kuiper belt (KB). The first KB object, 1992 QB_1, was found by Jewitt and Luu (*1*). To date, observations have yielded some 55 trans-Neptune bodies like 1992 QB_1 with radii on the order of 100 km or larger. Pluto is considered by some to be a member of this population (*2*). Based on the size of the sky area searched, estimates of the total population of such objects within 10° of the ecliptic is of the order of a few $\times\ 10^4$ (*3*). Smaller objects must also be present and the latest survey by Jewitt *et al.* (*4*) puts the total mass between 30 and 50 astronomical units (AU) at earth mass $0.26M_\oplus$ for objects greater than 1 km in diameter. Anderson *et al.* (*5*) have inferred a limit of $\sim 0.3M_\oplus$ out to 65 AU from the lack of penetration of the Pioneer 10's propellant tank by KB debris of radii > 0.5 cm. The discovery of $1996TL_{66}$ in a high eccentricity orbit points to an additional scattered KB component having a mass of $\sim 0.5M_\oplus$ with orbits between ~40 and 200 AU (*6*).

The dynamics of the KB is also the subject of considerable attention and has led to some indirect estimates of its mass. Residuals in the orbit of Halley's comet have been used to set a mass limit of order of $M_\oplus$ for a ring of comets at ~50 AU (*7*). Duncan *et al.* (*8*) demonstrated that a trans-Neptune disk could provide a plausible source for short-period comets. Test particle simulations appear to require an initial number density profile of $n(r) \sim 3 \times 10^6 \times (40\ \text{AU}/r)^2$ particles per AU^2 to account for the present flux of Jupiter family comets into the inner Solar System. This implies an initial surface density of $\sigma \sim 0.06(R/10\ \text{km})^3\ \text{g/cm}^2$ and a radial mass gradient $dM/dr \sim 2\pi\sigma r \sim 0.6(R/10\ \text{km})^3\ M_\oplus/\text{AU}$ at $r = 40$ AU. The bulk of these objects are probably comet sized with R equal to a few kilometers.

KB objects (KBOs) are far from uniform in their orbital characteristics, with most objects interior to ~40 AU observed to have high eccentricities and to reside in mean motion resonances with Neptune (*9*, *10*). These resonant orbits may be instrumental in preserving their occupants; it is well known that Pluto enjoys such protection through its 3:2 resonance that prevents close encounters with Neptune. On the other hand, numerical experiments show that many test particles achieve Neptune crossing status within 10^9 years because of the action of secular and mean motion resonances (*9*). Thus, the observed inner portion of the belt appears to be highly evolved, and, perhaps, substantially depleted from its primordial density. Extrapolating models of the planetesimal disk into this region implies that a few tens of earth masses originally may have resided there. A substantial disk of a few Neptune masses is required by the resonance sweeping hypothesis (*10*, *11*) in which the giant planets migrate, thereby trapping KBOs in low-order mean motion resonances. A massive primordial KB has also been postulated by Stern (*2*), who prefers ~10 to $50M_\oplus$ of material between 30 and 50 AU to collisionally assemble QB_1-type KBOs before Neptune's formation.

Here we introduce another constraint on the nature of the KB that has not yet been exploited. Galactic disks and planetary rings can transport angular momentum by spiral density waves, a process that can alter the orbital evolution of a perturbing body and change the structure of the disk (*12*). For example, many gaps and ringlets of Saturn's rings are due to perturbations from the planet's satellites (*13*), and simulations of planet-forming circumstellar gas disks indicate that density waves may result in the orbital decay of embedded protoplanets (*14*, *15*), which, in turn, may have relevance to the existence of close stellar companions (*16*). Neptune's eccentricity is curiously low ($e = 0.009$, where $e = 0$ for a circular orbit) compared with the other planets in the solar system.

Waves are excited at resonance sites, which are locations where a forcing frequency matches some natural frequency of the disk. The most numerous examples are mean motion resonances, where the disk's orbital period is commensurate with the perturber's orbital period. A less common situation involves the apse precession period of disk material as the natural frequency and the disk's wave response is referred to as an apsidal wave. These too have been detected in Saturn's rings, where the forcing frequency is the mean motion of Iapetus (*13*). Also included in this class is the secular resonance where the forcing frequency is the slow apsidal precession rate, ν_p, of the perturber (*17*, *18*). The resulting wavelengths are much longer than their mean-motion counterparts, and the wave's open spiral structure allows a perturber to gravitationally couple to a wide swath of the

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disk. As the wave propagates away from a resonance, the wavelength shortens (*19*) and the wave train takes on a spiral form (Fig. 1). This density perturbation is nonaxisymmetric, and the perturber's gravitational attraction for the density distribution gives rise to a torque (*20*). This is the type of resonance considered here.

Neptune's semimajor axis is a = 30.1 AU, its orbital period is $P = 2\pi/\Omega_p$ = 165 years, its mass ($M_N = 1.03 \times 10^{29}$ g) in solar units ($M_\odot = 2 \times 10^{33}$ g) is $\mu \equiv M_N/M_\odot = 5.15 \times 10^{-5}$. Neptune's precession frequency, ν_p = 0.673 arc sec/year, and its corresponding period, $P_\nu = 2\pi/\nu_p = 1.92 \times 10^6$ years, is largely due to the solar system's ν_8 eigenvector that dominates this planet's secular motions (*21*). The contribution of this eigenvector to its eccentricity is e_8 = 0.00912 (*22*). However, the generation of apsidal waves at Neptune's secular resonance tends to damp the planet's e (*18*), with the damping rate dependent on the perturbed wave portion of the disk's self-gravitational potential: $|\Phi_d| = \phi e\mu(a\Omega_p)^2$, where ϕ is a dimensionless strength coefficient that depends on the details of the disk's structure (*23*). It is also convenient to define $\mu_d \equiv \pi\sigma r^2/M_\odot$ as a nondimensional measure of the disk's surface density, and most of our results are couched in terms of this parameter. Because the mass of the disk within an annulus Δr is $\Delta M \sim 2\pi\sigma r\Delta r$, $\mu_d M_\odot$ is numerically equal to the mass contained within $\Delta r \sim r/2$.

In a KB that extends continuously beyond Neptune's orbit, the planet would excite apsidal waves at its secular resonance where the precession rate of disk material due to Neptune's gravity equals Neptune's precession rate due to the other planets. The waves propagate across the disk in regions where $\nu_p > \nu$, which is radially away from Neptune and deeper into the disk. Neptune's gravitational attraction on the spiral wave form gives rise to a torque $T = r|\Phi_d|^2/4G$ on the planet that causes its orbit to evolve. For $e \ll 1$, $\nu_p \ll \Omega_p$, it can be shown (*18, 20*) that the planet's orbital eccentricity varies at rate $\dot{e}/e \simeq -T/e^2a^2\Omega_p$ (*24*). This indicates that the generation of apsidal waves damps the planet's e (*25, 26*) with a characteristic time scale

$$\tau \equiv |e/\dot{e}| \approx 2P/(\phi^2\mu\pi) = 2 \times 10^6\phi^{-2} \text{ years} \quad (1)$$

If the small e of Neptune is a consequence of apsidal wave interaction with the KB, a minimum allowed e-folding time of $\tau \sim 10^9$ years is established by setting the initial e to unity and t equal to the age of the solar system (4.5×10^9 years). Hence, our hypothesis requires ϕ to be less than ϕ_c = 0.045 to avoid damping e below its observed value.

Assuming the primordial KB extended all the way inward to the orbit of Neptune, we find $\phi \approx |H_1^{(1)}(\Lambda)|/x_o$, where $\Lambda \equiv (\pi x_o)^{-1}(\mu/\mu_d)$, $x_o \equiv \Delta r_o/a = 0.309$, $\Delta r_o \equiv r_o - a = 9.3$ AU. $H_1^{(1)}$ denotes a Hankel function (*27*). At large Λ (low disk mass), its asymptotic form is $\sqrt{2/\pi\Lambda}\,\exp i(\Lambda - 3\pi/4)$ so that $\phi \rightarrow \sqrt{2\mu_d/x_o\mu}$. To account for Neptune's finite e, such a disk would have to have $\mu_d \lesssim \mu x_0\phi_c^2/2 = 1.6 \times 10^{-8}$. This is equivalent to 0.0054 $M_\oplus$ ($\mu_\oplus \equiv M_\oplus/M_\odot = 3 \times 10^{-6}$), which is about two orders of magnitude below current estimates of the mass of the observable region of the KB.

On the other hand, observations of the KB as far as ~45 AU (*3*) indicate that the current KB has random velocities too high to sustain wave action (see below). The structure of this low-density region is similar to results from long-term test particle integrations of the outer solar system, implying that the giant planets' gravitational perturbations acting over the age of the system tend to destabilize orbits of bodies in the inner part of the KB (*9*). Accordingly, Neptune may have been able to erode the KB out to the vicinity of its 2:1 mean motion resonance at ~48 AU before its orbit could be circularized (*28*). However, any disk material orbiting beyond 48 AU would retain the bulk of its primordial mass. This has led to the suggestion that the surface density in this region may rise significantly (*2*), approaching more closely that believed to have existed in Neptune's zone before its formation—$\sigma \sim O(10^{-1})$ g/cm^2—implying $\mu_d \sim O(\mu)$.

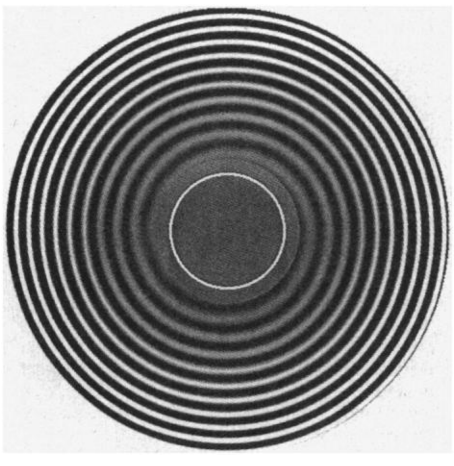

Fig. 1. Surface density for a one-armed apsidal wave excited by Neptune in a trans-Neptune disk with nondimensional mass parameter $\mu_d = \pi\sigma r^2/M_\odot = 2M_\oplus/M_\odot$. White circle indicates Neptune's orbit at 30.1 AU. Crests of the waves are white; the gray scale is stretched to reveal the contrast. Surface density is assumed to vary as r^{-2}, so the wavelength decreases monotonically.

Adopting a KB structure that has been depleted in material interior to a distance of ~48 AU results in lower amplitude waves that are initiated from the disk's inner boundary r_e. The behavior of ϕ is shown for the case $\Lambda = 1$ (disk mass comparable to Neptune) as the inner edge of the disk moves outward (Fig. 2). As the edge moves beyond the resonance, the torque drops. Nevertheless, the critical value ϕ_c = 0.045 requires an edge far outside Neptune resonance sites, where there is no known source of erosion. Thus, it is difficult to reconcile a massive KB with Neptune's e. For other KB masses, an approximate relationship between the allowed mass of the disk and the location of its edge, $\Delta r_e \equiv r_e - a$ can be found (*29*)

$$\mu_d \approx \frac{1}{2}\phi_c\mu\left(\frac{\Delta r_e}{\Delta r_o}\right)^2 \approx 1.2 \times 10^{-6}\left(\frac{\Delta r_e}{\Delta r_o}\right)^2 \quad (2)$$

If the disk edge lies in the vicinity of the outer 2:1 mean motion resonance, $\Delta r_e \sim 1.91\Delta r_o$, and $\mu_d \sim 4.2 \times 10^{-6}$, which is just over $M_\oplus$. Setting r_e = 48 AU as the likely distance out to which the disk could be eroded, numerically integrated values of ϕ and the corresponding characteristic time scale, τ, are shown in Fig. 3 as functions of μ_d. In making these calculations, ϕ has been numerically integrated without approximation. A time scale of $\tau \sim 10^9$ years implies an upper limit of $\mu_d \sim 8 \times 10^{-6}$. Placing a Neptune mass in the disk decreases the characteristic time scale by an order of magnitude implying $e_8 \lesssim 10^{-20}$ after 4.5×10^9 years.

These findings are sensitive to the kinetic state of the outer KB that provides the medium for wave propagation. One criterion for density wave action is that there

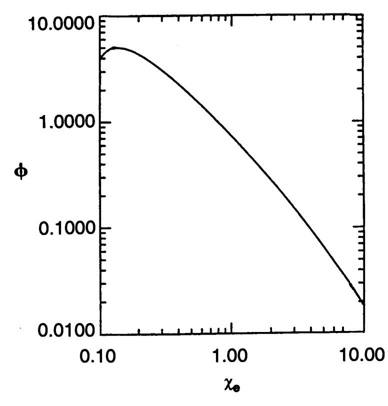

Fig. 2. Value of the strength coefficient $\phi \equiv |\Phi_d|/[e\mu(a\Omega)^2]$ versus the ratio $\chi_e = \Delta r_e/\Delta r_o$ of the planet-edge separation over the planet-resonance separation. The value of $\Lambda \equiv (\pi x_o)^{-1}(\mu/\mu_d)$ has been set to unity. The disk edge would lie at resonance for $\chi_e = 1$; the coefficient drops with $1/\chi_e^2$ once the disk edge is beyond the resonance site.

must be numerous disk particles distributed across each wave cycle or, equivalently, that the number density of particles, σ/M_p exceeds $1/\lambda^2$, where $\lambda = 2\pi/k$ is the wavelength and $M_p = 4\pi\rho R^3/3$ is the characteristic particle mass having body density $\rho \sim 2$ g/cm^3 and radius R. In terms of the particle radius, this limit becomes $R < \mu_d(3M_\odot\Omega^2/\rho\nu_p^2)^{1/3} = 2.2 \times 10^3$ $(\mu_d/\mu_\oplus)$ km. A more restrictive requirement is that the dispersion velocities of the particles should not cause radial excursions in excess of the wavelength. Assuming a dispersion velocity comparable to the escape velocity from particle sizes contributing most of the mass, this results in constraint $R < 0.16\mu_d r\Omega^2/\nu_p\sqrt{G\rho}$ ~ 72 $(\mu_d/\mu_\oplus)$ km. Even though the known KBOs have sizes $R \sim 100$ km, the bulk of the disk's mass may be contained in comet-sized bodies having R of a few kilometers, so the size criterion would be satisfied (*30*). However, if large objects ($R \gg 10^2$ km) exist in the KB in sufficient numbers to pump up dispersion velocities (*31*), a hot disk ($\nu > \Omega/k$) will not sustain waves efficiently, and a larger mass could be accommodated. In this case, our model establishes a lower limit on the allowed dispersion velocity of the disk versus its mass: $V \sim \Omega/k = 0.3(\mu_d/\mu)r\Omega$.

If the distant KB beyond ~48 AU is a cool disk ($\nu \ll \Omega/k$) (*32*) composed predominantly of comet-sized bodies (kilometers), Neptune could have launched apsidal waves of sufficient strength to damp its eccentricity to its observed small value of $e_8 = 0.009$. This implies an upper limit of the KB mass out to ~75 AU on the order of 2 $M_\oplus$ (Fig. 3). In contrast, larger values (comparable to the mass of Neptune) would have damped e_8 in only O(10^8) years. Finally, we should point out that the upper limit was established by the fact that the primordial e could not exceed unity. If we adopt a more realistic starting value of $e_8 \sim 0.05$ for Neptune's primordial eccentricity, the minimum characteristic time scale increases to $\tau \sim 2.6 \times 10^9$ years, the upper limit of the KB mass out to $r + \Delta r \sim 75$ AU is $\mu_d \sim 5 \times 10^{-6}$ or ~1.6 $M_\oplus$ (Fig. 3).

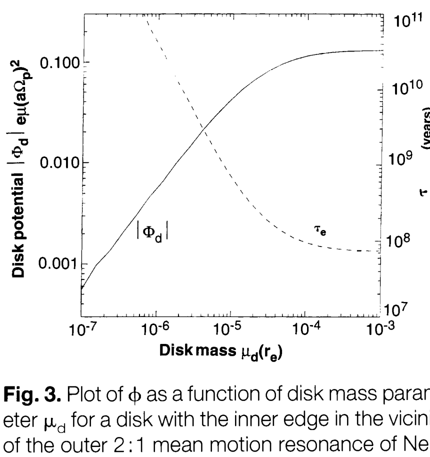

Fig. 3. Plot of ϕ as a function of disk mass parameter μ_d for a disk with the inner edge in the vicinity of the outer 2:1 mean motion resonance of Neptune, $r_e \simeq 48$ AU. Also shown is the characteristic damping time τ (dashed curve). A mass value comparable to Neptune, $\mu_d \sim 5.15 \times 10^{-5}$, leads to a time scale of only $\tau \sim 10^8$ years. A characteristic time of $\tau = 2.6 \times 10^9$ years, necessary to decay an initial eccentricity of $e \sim 0.05$ to its current value within the age of the solar system, corresponds to $\mu_d \sim 5 \times 10^{-6}$, which is the equivalent of ~$1.6M_\oplus$.

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17. The apsidal line runs through the periapse of the orbit, and gravitational perturbations from the other planets cause it to precess in space in a counterclockwise manner.
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19. The dispersion relationship for the waves is $2\pi G\sigma|k| = \kappa^2 - m(\Omega - \Omega_{ps})^2 \equiv D$, where k is the wave number, κ is the disk's epicycle frequency, m is the number of spiral arms, and Ω_{ps} is the angular speed at which the spiral pattern rotates, $\Omega = \sqrt{GM_\odot/r^3}$ is the disk mean motion, G is the gravitational constant, and ν_p and ν are the perihelia precession rates of the planet and disk, respectively. For apsidal waves, $m = 1$, $\Omega_{ps} = \nu_p$, and the frequency distance from resonance $D = (\kappa + \Omega - \nu_p)(\kappa - \Omega + \nu_p)$. The apse precession rate is given by $\nu = \Omega - \kappa \ll \Omega$ so that $D \approx 2\Omega(\nu_p - \nu)$, which leads to $|k| \approx (\nu_p - \nu)\Omega/\pi G\sigma$.
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22. More precisely, the secular motions of the planets are described by a superposition of eigenvectors with eigenfrequencies $\{\nu_j\}$. We are concentrating on the eigenvector associated with the ν_8 frequency, which has a contribution to Neptune's e of $e_8 = 0.00912$ (*21*). Once the ν_8 contribution damps out, Neptune's e would be dominated by the ν_7 eigenfrequency (mostly driven by Uranus) with an amplitude of $e \sim 0.0037$.
23. The coefficient ϕ is given by $\phi = |\int_{r_e}^{\infty}(1/2)b_{3/2}^{(2)}(r/a)\exp(i\int^r k(r')dr')dr/a|$ (*18*) where r is the heliocentric distance, r_e is the radius of the disk's inner edge, and $b_s^{(m)}(\beta) \equiv (2/\pi)\int_0^\pi \cos(m\theta)(1 - 2\beta\cos\theta + \beta^2)^{-s}d\theta$ is a Laplace coefficient [D. Brouwer and G. M. Clemence, *Methods of Celestial Mechanics* (Academic Press, New York, 1964)]. If m, $(\beta - 1) \ll 1$, the integration can be simplified because Laplace coefficients can be approximated $b_{3/2}^{(m)} \approx 2/[\pi\sqrt{\beta}(\beta - 1)^2]$, which is independent of m. In this case, the precession rate is $\nu(\beta) \sim \mu\Omega_p/2\pi(\beta - 1)^2$. We now define $x \equiv \beta - 1$ as the nondimensional distance from Neptune but set $\beta \simeq 1$ when it does not appear in this combination. The nondimensional distance of the resonance site is $\Delta r_o/a \equiv x_o \approx \sqrt{\mu\Omega_p/2\pi\nu_p} = 0.309$, and the resonance location is $r_o = a(1 + x_o) = 39.4$ AU. The wavenumber satisfies $rk \approx (2\pi)^{-1}(\mu/\mu_d)(x_o^{-2} - x^{-2})$, and ϕ can be approximated by $\phi \approx (\pi x_o)^{-1}|\int_{\chi_e}^{\infty}\chi^{-2}e^{i\Lambda(\chi+\chi^{-1})/2}d\chi|$ where $\chi \equiv x/x_o$, $\chi_e \equiv x_e/x_o$ and $\Lambda \equiv (\pi x_o)^{-1}(\mu/\mu_d)$.
24. This follows from the Jacobi integral. Because the spiral pattern rotates at a constant angular velocity, ν_p, the planet's Jacobi integral $J = E - \nu_p L$ is conserved, where $E = -GM_\odot/2a$ and $L = \sqrt{GM_\odot a(1 - e^2)}$ are the planet's specific energy and angular momentum, respectively.
25. If the planet instead launched waves at a secular resonance interior to its orbit, its eccentricity would increase.
26. There is also a small expansion of the orbit; although driving Neptune's e to zero from its present value of ~10^{-2} would expand the planet's orbit by only $\Delta a/a \simeq e^2\nu_p/\Omega_p \sim 10^{-8}$.
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28. Because the accretion time scale for Neptune is very long (a few years $\times 10^8$), impacts may continuously generate some eccentricity. Hence, this time scale, rather than the eccentricity damping time given by equation 1, sets the upper limit for disk erosion.
29. The $1/\chi$ term in the exponent can be dropped, and the integral of (*23*) can be integrated by parts to find $\phi = (\Lambda/2\pi x_o)|e^{iz}/z + iE_1(-iz)| = (\Lambda/2\pi x_o)\sqrt{(f(z) - z^{-1})^2 + g(z)^2}$, where $z \equiv \chi_e\Lambda/2$, E_1 is the exponential integral, and f and g are its nonoscillatory auxiliary functions (*27*). At large z, auxiliary functions have the asymptotic forms $f(z) \approx 1/z - 2/z^3 + O(z^{-5})$, $g(z) \approx 1/z^2 + O(z^{-4})$, and $\phi \approx \Lambda/2\pi x_o z_e^2 = (2\mu_d/\mu)(x_o/x_e)^2$.
30. The size distribution among the observed KBOs varies as $\propto R^{-4}$ so most of the disk's mass is concentrated among its smallest members (*4*). The KB is also the purported source of the short-period comets (*8*), which have R sizes of a few kilometers. These two lines of evidence indicate that most of the KB's mass is contained in small, kilometer-sized bodies.
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32. A scattered disk of high eccentricity objects placed there by Neptune (*6*) is still possible because such a population would not support wave action. However, it would not represent a remnant of material originally comprising a trans-Neptune disk.

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