MATHEMATICS

Fractions to Make an Egyptian Scribe Blanch

Over 3500 years ago, the Egyptian scribe Ahmes wrote a scroll of mathematical problems intended to instruct his readers in the "knowledge of all obscure secrets"—including how to multiply and divide fractions. The scroll, called the Rhind Papyrus, is also a primer on the Egyptian way of writing fractions: as sums of reciprocals of whole numbers, using as few numbers as possible, and always avoiding repetition. Thus, Ahmes wrote the number we call "2/17" as 1/12 + 1/51 + 1/68. Bizarre as it looks, this method gave exact results with a surprising economy of notation, and the Rhind Papyrus showed how to use it to solve a number of supposedly practical problems involving the division of loaves and beer.

Now, two mathematicians have given the ancient method of Egyptian fractions a perverse modern twist. For the sheer delight of it, they have shown how run-of-the-mill numbers can be expressed as fractions so sprawling that they would have given the ancient scribe carpal tunnel syndrome—if he didn't run out of papyrus first.

Classically, Egyptian fractions were very sparse, involving reciprocals of widely spaced numbers. But Greg Martin, a number theorist at the Institute for Advanced Study in Princeton, New Jersey, went to the opposite extreme by asking: How densely spaced can the terms in an Egyptian fraction get? Or, to put it another way, if we want to divide r kegs of beer among our friends in the Egyptian way, so that they all get different amounts but nobody gets less than a certain amount—$1/n$ kegs—how many friends can we serve?

According to Martin, we can serve a lot more friends than Ahmes, or even modern mathematicians, would have suspected: Two kegs of beer can be divided among 366 people, with everyone getting at least 1/1000 of a keg and no two people getting the same amount. Reduce the limit to 1/3,000,000 of a keg, and nearly 1 million people could get some minute fraction of the beer.

A string of fractions that long would ordinarily add up not to an integer like 2, but to a fraction with a gigantic denominator—the least common denominator of all the summands. "We've all seen examples in fifth-grade arithmetic where the denominator gets bigger—and the more fractions we add, the bigger and messier it gets," Martin says. "To add a million fractions together and get a denominator of 1 … is just outside our experience."

The key to creating a dense Egyptian fraction that can add up to an integer is choosing the denominators of the reciprocals, or unit fractions, so that you can do a spectacular amount of cancellation. And as Martin showed in a paper presented at a number theory conference this summer at Pennsylvania State University, such cancellation can always be arranged, provided that n, the denominator of the smallest allowable fraction, is large enough, and provided the number of fractions doesn't exceed about 30% of n.

It's just a matter of smoothness—not of the beer, but of the fractions. "Smooth" numbers, a term coined by Ronald Rivest of the Massachusetts Institute of Technology, are numbers with no large prime divisors. They are an essential ingredient in every state-of-the-art computer factorization algorithm, but for Martin's purposes, the most important fact is that there are lots of them: For any number n, roughly 30% of the numbers between 1 and n have no prime factors greater than the square root of n. These are, with a few strategic deletions, precisely the numbers Martin used as the denominators of his unit fractions. Because they inevitably have many common factors, they open the way to the extensive cancellation needed to get a denominator of 1.

$$2=\frac{1}{13}+\frac{1}{19}+\frac{1}{23}+\frac{1}{26}+\frac{1}{27}+\frac{1}{29}+\frac{1}{32}+\frac{1}{34}+\frac{1}{35}+\frac{1}{36}+\frac{1}{38}+\frac{1}{39}+\frac{1}{40}+\frac{1}{42}+\frac{1}{44}+\frac{1}{45}+\frac{1}{46}+\frac{1}{49}+\frac{1}{50}+\frac{1}{51}+$$
$$\frac{1}{52}+\frac{1}{55}+\frac{1}{56}+\frac{1}{57}+\frac{1}{58}+\frac{1}{60}+\frac{1}{62}+\frac{1}{63}+\frac{1}{64}+\frac{1}{65}+\frac{1}{66}+\frac{1}{68}+\frac{1}{69}+\frac{1}{70}+\frac{1}{72}+\frac{1}{75}+\frac{1}{76}+\frac{1}{77}+\frac{1}{78}+\frac{1}{80}+\frac{1}{81}+$$
$$\frac{1}{84}+\frac{1}{85}+\frac{1}{87}+\frac{1}{90}+\frac{1}{91}+\frac{1}{92}+\frac{1}{93}+\frac{1}{95}+\frac{1}{96}+\frac{1}{98}+\frac{1}{99}+\frac{1}{102}+\frac{1}{104}+\frac{1}{105}+\frac{1}{108}+\frac{1}{110}+\frac{1}{112}+\;.\;.\;.$$
$$\frac{1}{870}+\frac{1}{874}+\frac{1}{875}+\frac{1}{880}+\frac{1}{882}+\frac{1}{884}+\frac{1}{891}+\frac{1}{896}+\frac{1}{897}+\frac{1}{899}+\frac{1}{900}+\frac{1}{910}+\frac{1}{912}+\frac{1}{918}+\frac{1}{920}+\frac{1}{924}+\frac{1}{928}+$$
$$\frac{1}{930}+\frac{1}{931}+\frac{1}{935}+\frac{1}{936}+\frac{1}{945}+\frac{1}{950}+\frac{1}{952}+\frac{1}{957}+\frac{1}{960}+\frac{1}{966}+\frac{1}{969}+\frac{1}{975}+\frac{1}{980}+\frac{1}{986}+\frac{1}{988}+\frac{1}{990}+\frac{1}{992}$$

Number of terms: 366

SOURCE: G. MARTIN

Thinly sliced. Some of the 366 terms in the modern Egyptian fraction that adds up to 2. Ancient versions of these sums of reciprocals were far more compact.

Other number theorists are persuaded by the finding, and another young mathematician recently proved a remarkably similar result. Ernest Croot III, a graduate student at the University of Georgia in Athens, has reportedly solved a question raised by the great mathematician Paul Erdös, who died last September (*Science*, 7 February, p. 759). Numbers theorists have long known that the unit fractions from 1 to $1/n$ add up to a number roughly equal to the natural logarithm of n plus "Euler's constant," 0.577… . For example, 1 + 1/2 + … + 1/1000 is about 7.5. Thus, 8 is too large to be represented as a sum of unit fractions between 1 and 1/1000, but conceivably some of the numbers from 1 through 7 could be. Erdös asked what percentage of integers less than the logarithm of n could be written as an Egyptian fraction without using any fractions beyond $1/n$. The very wording of the question suggests that he doubted they all could.

Croot, however, has shown that—at least for large enough values of n—Erdös was too pessimistic. In fact, every integer at least 0.2 less than the number $1 + 1/2 + \ldots + 1/n$ can be represented this way. If $n = 1000$ is sufficiently large (Croot isn't sure it is), then all the integers from 1 to 7 would be expressible as Egyptian fractions with no denominators greater than 1000. Like Martin's result, Croot's proof implies the existence of extraordinarily dense fractions. To get a sum that's close to the maximum possible (log n plus Euler's constant), you've got to use most of the terms available to you.

Croot's work is still unpublished, but Andrew Granville of the University of Georgia, a leading number theorist and Croot's thesis adviser, has read the proof and says it is "certainly correct." Granville adds that it is "sheer coincidence" that Croot and Martin happened to get interested in Egyptian fractions at the same time. "Ernie tends to work on whatever takes his fancy—he'll just pop into my office and show me some ingenious thing I didn't even know he was working on."

–Dana Mackenzie

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