Energy Conditions in the Epoch of Galaxy Formation

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The energy conditions of Einsteinian gravity (classical general relativity) do not require one to fix a specific equation of state. In a Friedmann-Robertson-Walker universe where the equation of state for the cosmological fluid is uncertain, the energy conditions provide simple, model-independent, and robust bounds on the behavior of the density and look-back time as a function of red shift. Current observations suggest that the "strong energy condition" was violated sometime between the epoch of galaxy formation and the present. This implies that no possible combination of "normal" matter is capable of fitting the observational data.

The energy conditions of Einsteinian gravity (classical general relativity) place restrictions on the stress-energy tensor $T_{\mu\nu}$ (energy-momentum tensor) (*1–3*). This tensor is a 4-by-4 matrix built up out of the energy density, momentum density, and the 3-by-3 stress tensor (pressure and anisotropic stresses). The energy conditions force various linear combinations of these quantities to be positive and have been used, for instance, to derive many theorems of classical general relativity—such as the singularity theorems, the area increase theorem for black holes, and the positive mass theorem—without the need to assume a specific equation of state (*4–6*). General relativists and particle physicists would be surprised if large violations of the classical energy conditions occur at temperatures significantly below the Planck scale $kT < E_{\text{Planck}} \approx 10^{19}$ GeV, $T \approx 10^{32}$ K (*7*). [Above the Planck scale, quantum gravity takes over, the whole framework of classical cosmology seems to break down, and the question is moot (*8–10*).]

Current observations seem to indicate that the "strong energy condition" (SEC) is violated rather late in the life of the universe—somewhere between galaxy formation and the present time, in an epoch in which the cosmological temperature never exceeds 60 K. Here I show this by using the energy conditions to develop simple and robust bounds for the density and look-back time (*11*) as a function of red shift in a Friedmann-Robertson-Walker (FRW) cosmology. The experimental observations I need are the present-day value of the Hubble parameter H_0, an estimate for the age of the oldest stars in the galactic halo, and an estimate for the red shift at which these oldest stars formed. From the theoretical side, I only need to use an FRW cosmology that is subject to the Einstein equations and classical energy conditions. Using the energy conditions to place bounds on physical parameters of the universe allows me to avoid the need to separately analyze cold, hot, lukewarm, or mixed dark matter. Similarly, MACHOS (massive compact halo objects), WIMPS (weakly interacting massive particles), axions, massive neutrinos, and other hypothetical contributions to the cosmological density are automatically included as special cases of this analysis.

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The bounds I derive from the SEC are independent of whether or not the universe is open, flat, or closed, which means that the density parameter (Ω parameter, the ratio of the actual density to the critical density needed to close the universe) does not have to be specified. Thus, my approach is independent of the existence or nonexistence of any of the standard variants of cosmological inflation (*12*), which typically predict $\Omega = 1$ (*13–15*).

If current observations are correct, then the SEC must be violated sometime between the epoch of galaxy formation and the present. This implies that no possible combination of "normal" matter is capable of fitting the observational data, and one needs to do something drastic to the cosmological fluid—either introduce a cosmological constant Λ (*16*) or have a very nonstandard weak form of cosmological inflation that persists right up to galaxy formation.

The space-time geometry of the standard FRW cosmology is described by specifying the geometry of space as a function of time, using the space-time metric (*17, 18*)

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 \times (d\theta^2 + \sin^2\theta d\phi^2)\right] \quad (1)$$

Here ds is the invariant interval between two events, t is co-moving time (time as measured by an observer following the average Hubble flow), $a(t)$ is the scale parameter describing the size of the universe as a function of time, and r, θ, and ϕ are spherical polar coordinates used to cover all of space (with space at time t being defined as the constant-t slice through the space-time). The parameter k depends on the overall geometry of space, and only takes on the values

$$k = \begin{cases} +1 & \text{closed} \quad (\text{if } \Omega > 1), \\ 0 & \text{flat} \quad (\text{if } \Omega = 1), \\ -1 & \text{open} \quad (\text{if } \Omega < 1). \end{cases} \quad (2)$$

The two nontrivial components of the Einstein equations yield the total density ρ and total pressure p of the cosmological fluid as a function of the scale factor a and Newton's constant G (*19*).

$$\rho = \frac{3}{8\pi G}\left[\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right] \quad (3)$$

$$p = -\frac{1}{8\pi G}\left[2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right] \quad (4)$$

They can be combined to deduce the conservation of stress energy

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) \quad (5)$$

Here $\dot{a}$ is the (time-dependent) velocity of expansion of the universe. Combined with the scale factor $a(t)$, it defines the (time-dependent) Hubble parameter

$$H(t) = \frac{\dot{a}(t)}{a(t)} \quad (6)$$

There are several different types of energy condition in general relativity, the two main classes being averaged energy conditions (that depend on some average of the stress-energy tensor along a suitable curve) and the pointwise energy conditions (that depend only on the stress-energy tensor at a given point in spacetime). The standard pointwise energy conditions are the null energy condition (NEC), weak energy condition (WEC), SEC, and dominant energy condition (DEC). Basic definitions are given in (*1–3*) and for the special case of an FRW space-time, the general formulas simplify.

$$\text{NEC} \Leftrightarrow (\rho + p \geq 0) \quad (7)$$

$$\text{WEC} \Leftrightarrow (\rho \geq 0) \text{ and } (\rho + p \geq 0) \quad (8)$$

$$\text{SEC} \Leftrightarrow (\rho + 3p \geq 0) \text{ and } (\rho + p \geq 0) \quad (9)$$

$$\text{DEC} \Leftrightarrow (\rho \geq 0) \text{ and } (\rho \pm p \geq 0) \quad (10)$$

The NEC is enough to guarantee that the density of the universe goes down as its size increases.

$$\text{NEC} \Leftrightarrow \text{sign}(\dot{\rho}) = -\text{sign}(\dot{a}) \quad (11)$$

If the NEC is violated, the density of the universe must increase as the universe expands, so something has gone very seriously wrong. The WEC additionally requires that the density be positive (*20*).

To understand what the SEC requires for the physical universe, consider the quantity $d(\rho a^2)/dt$ and use the Einstein equations to deduce

$$\frac{d}{dt}(\rho a^2) = -a\dot{a}(\rho + 3p) \qquad (12)$$

Thus

$$\text{SEC} \Rightarrow \text{sign}\left[\frac{d}{dt}(\rho a^2)\right] = -\text{sign}(\dot{a}) \qquad (13)$$

This implies that

$$\text{SEC} \Rightarrow \rho(a) \geq \rho_0(a_0/a)^2 \text{ for } a < a_0 \qquad (14)$$

In terms of the red shift $(1 + z = a_0/a)$:

$$\text{SEC} \Rightarrow \rho(z) \geq \rho_0(1 + z)^2 \qquad (15)$$

The subscript zero denotes present-day values, and the SEC provides a model-independent lower bound on the density of the universe extrapolated back to the time of the Big Bang. Another viewpoint on the SEC comes from considering the quantity

$$\rho + 3p = -\frac{3}{4\pi G}\left[\frac{\ddot{a}}{a}\right] \qquad (16)$$

That is

$$\text{SEC} \Rightarrow \ddot{a} < 0 \qquad (17)$$

The SEC implies that the expansion of the universe is decelerating, and this conclusion holds independently of whether the universe is open, flat, or closed.

For the DEC, use the Einstein equations to compute

$$\frac{d}{dt}(\rho a^6) = +\,3a^5\dot{a}(\rho - p) \qquad (18)$$

Thus

$$\text{DEC} \Rightarrow \text{sign}\left[\frac{d}{dt}(\rho a^6)\right] = +\,\text{sign}(\dot{a}) \qquad (19)$$

The DEC therefore provides an upper bound on the energy density.

$$\text{DEC} \Rightarrow \rho(a) \leq \rho_0(a_0/a)^6 \text{ for } a < a_0 \qquad (20)$$

In terms of the red shift

$$\text{DEC} \Rightarrow \rho(z) \leq \rho_0(1 + z)^6 \qquad (21)$$

When we look into the sky and see some object, the look-back time to that object ($\tau = t_0 - t$) is defined as the difference between t_0 (the age of the universe now) and t (the age of the universe when the light that we are now receiving was emitted). If we know the velocity of expansion of the universe $\dot{a}$ as a function of scale parameter a we simply have

$$\tau(a; a_0) = t_0 - t = \int_a^{a_0} \frac{da}{\dot{a}(a)} \qquad (22)$$

By putting a lower bound on $\dot{a}$ we deduce an upper bound on look-back time. In particular because the SEC implies that the expansion is decelerating, then

$$\text{SEC} \Rightarrow \tau(a; a_0) = t_0 - t \leq \frac{1}{H_0}\frac{a_0 - a}{a_0} \qquad (23)$$

independent of whether the universe is open, flat, or closed. Expressed in terms of the red shift

$$\text{SEC} \Rightarrow \tau(z) = t_0 - t \leq \frac{1}{H_0}\frac{z}{1 + z} \qquad (24)$$

This provides us with a robust upper bound on the Hubble parameter

$$\text{SEC} \Rightarrow \forall z : H_0 \leq \frac{1}{\tau(z)}\frac{z}{1 + z} \qquad (25)$$

This is enough to illustrate the age-of-the-oldest-stars problem (often mischaracterized as the age-of-the-universe problem). Suppose there is some class of standard candles whose age of formation, τ_f, can be estimated (*21*). Suppose further that if we look out far enough we can see some of these standard candles forming at red shift z_f (or can estimate the red shift at formation). Then

$$\text{SEC} \Rightarrow H_0 \leq \frac{1}{\tau_f}\frac{z_f}{1 + z_f} \leq \frac{1}{\tau_f} \qquad (26)$$

The standard candles currently of most interest (simply because they have the best available data and provide the strongest limit) are the globular clusters in the halos of spiral galaxies: Stellar evolution models estimate (they do not measure) the age of the oldest stars still extant to be $16 \pm 2 \times 10^9$ years (*22*). That is, at an absolute minimum

$$\text{Age of oldest stars} \equiv \tau_f \geq 16 \pm 2 \times 10^9 \text{ years} \qquad (27)$$

Using $z_f < \infty$, this implies that (*23*)

$$H_0 \leq \tau_f^{-1} \leq 62 \pm 8 \text{ km s}^{-1}\,\text{Mpc}^{-1} \qquad (28)$$

When we actually look into the night sky, we infer that the oldest stars seem to have formed somewhat earlier than the development of galactic spiral structure (*24*). A canonical first estimate is (*25*)

$$\text{Red shift at formation of oldest } stars \equiv z_f \approx 15 \qquad (29)$$

This now bounds the Hubble parameter

$$\text{SEC} \Rightarrow H_0 \leq 58 \pm 7 \text{ km s}^{-1}\,\text{Mpc}^{-1} \qquad (30)$$

Recent estimates of the present-day value of the Hubble parameter are (*26*)

$$H_0 \in (65, 85) \text{ km s}^{-1}\,\text{Mpc}^{-1} \qquad (31)$$

[I have chosen to use a range of H_0 values on which there is widespread though not universal consensus (*26*).] But even the lowest reasonable value, $H_0 = 65$ km s^{-1} Mpc^{-1}, is only just barely compatible with the SEC and that only by taking the youngest reasonable value for the age of the globular clusters. For currently favored values of H_0, we deduce that the SEC must be violated somewhere between the formation of the oldest stars and the present time.

The following qualifications should be attached to this claim: We have to rely on both stellar structure calculations for τ_f and an estimate for z_f. Decreasing z_f to be more in line with the formation of the rest of the galactic structure ($z_f \approx 7$) makes the problem worse, not better ($H_0 \leq 54 \pm 7$ km s^{-1} Mpc^{-1}). Increasing z_f out to its maximum conceivable value $z_f \approx 20$ (*24*) does not greatly improve the fit to the SEC because the bound becomes $H_0 \leq 59 \pm 8$ km s^{-1} Mpc^{-1}. All of these difficulties are occurring at low cosmological temperatures ($T \leq 60$ K) and late times, in a region where the basic equation of state of the cosmological fluid is supposedly understood (*27*).

In contrast, the NEC does not provide any strong constraint on H_0. For a spatially flat universe [$k = 0$, $\Omega = 1$, as preferred by inflation advocates (*13–15*)]

$$\text{NEC} + (k = 0) \Rightarrow \tau = t_0 - t \leq \frac{\ln(1 + z)}{H_0} \qquad (32)$$

Somewhat more complicated formulas can be derived for $k = \pm 1$ (open or closed universes).

This implies a (very weak) bound on H_0. In order for cosmological expansion to be compatible with stellar evolution and the NEC

$$\text{NEC} + (k = 0) \Rightarrow H_0 \leq \frac{\ln(1 + z_f)}{\tau_f} \qquad (33)$$

The best value for τ_f (16×10^9 years) and best guess for z_f ($z_f \approx 15$) give $H_0 \leq 170$ km s^{-1} Mpc^{-1}. Decreasing z_f to about 7 reduces this bound slightly to $H_0 \leq 129$ km s^{-1} Mpc^{-1}. Both of these values are consistent with the observational bounds on H_0. Even for the highest Hubble parameter ($H_0 = 85$ km s^{-1} Mpc^{-1}) and oldest age for the oldest stars ($t_f = 18 \times 10^9$ years), $z_f \geq 3.6$, which is well within the observational bounds on z_f. The present data are therefore not in conflict with the NEC.

The DEC provides us with an upper bound on the energy density ρ and therefore an upper bound on the rate of expansion. This translates to a lower bound on the look-back time and a lower bound on the Hubble parameter. For a spatially flat universe

$$\text{DEC} + (k = 0) \Rightarrow \tau = t_0 - t \geq \frac{1}{3H_0}\frac{a_0^3 - a^3}{a_0^3} \qquad (34)$$

Somewhat more complicated formulas can be derived for $k = \pm 1$. In terms of the red shift

$$\text{DEC} + (k = 0) \Rightarrow \tau = t_0 - t \geq \frac{1}{3H_0}\left(1 - \frac{1}{(1+z)^3}\right) \quad (35)$$

So the ages of the oldest stars provide the constraint

$$\text{DEC} + (k = 0) \Rightarrow H_0 \geq \frac{1}{3\tau_f}\left(1 - \frac{1}{(1+z_f)^3}\right) \quad (36)$$

This also is a relatively weak constraint,

$$\text{DEC} + (k = 0) \Rightarrow H_0 \geq 20 \pm 3 \text{ km s}^{-1}\,\text{Mpc}^{-1} \quad (37)$$

The present observational data are also not in conflict with the DEC.

The estimated value of H_0 has historically exhibited considerable flexibility. Although it is clear that the relation between the distance and the red shift is essentially linear, the absolute calibration of the slope of the Hubble diagram (velocity of recession versus distance) has varied by more than an order of magnitude over the course of this century. Hubble parameter estimates from 500 km s^{-1} Mpc^{-1} to 25 km s^{-1} Mpc^{-1} can be found in the published literature (*28*, *29*). Current measurements give credence to the range 65 to 85 km s^{-1} Mpc^{-1} (*26*). The reliability of the data on τ_f and z_f is harder to quantify, but there appears to be broad consensus within the community on these values (*22*, *24*, *25*).

If the SEC is violated between the epoch of galaxy formation and the present, how does this affect our ideas concerning the evolution of the universe? The two favorite ways of allowing SEC violations in a classical field theory are by using a massive (or self-interacting) scalar field (*30*) or by using a positive cosmological constant (*31*). A classical scalar field φ that interacts with itself via some scalar potential $V(\varphi)$ can violate the SEC (*30*) but not the NEC, WEC, and DEC (*31*). Indeed

$$(\rho + 3p)|_{\varphi} = \dot{\varphi}^2 - V(\varphi) \quad (38)$$

It is this potential violation of the SEC (depending on the details of the time rate of change of the scalar field and its self-interaction potential) that makes cosmological scalar fields so attractive to advocates of inflation (*13–15*). In the present context, using a massive scalar field to deal with the age-of-the-oldest-stars problem is tantamount to asserting that a last dying gasp of inflation took place as the galaxies were being formed. This is viewed as an unlikely scenario (*32*).

In contrast, the current favorite fix for the age-of-the-oldest-stars problem is to introduce a positive cosmological constant Λ (*16*), in which case

$$(\rho + 3p)_{\text{total}} = (\rho + 3p)_{\text{normal}} - 2\rho_\Lambda \quad (39)$$

The observed SEC violations then imply

$$\rho_\Lambda \geq \frac{1}{2}(\rho + 3p)_{\text{normal}} \quad (40)$$

Under the mild constraint that the pressure due to normal matter in the present epoch be positive, this implies that more that 33% of the present-day energy density is due to a cosmological constant.

I have shown that high values of H_0 imply that the SEC must be violated sometime between the epoch of galaxy formation and the present. This implies that the age-of-the-oldest-stars problem cannot simply be fixed by adjusting the equation of state of the cosmological fluid. Because all normal matter satisfies the SEC, fixing the age-of-the-oldest-stars problem will inescapably require the introduction of "abnormal" matter; indeed, large quantities of abnormal matter, sufficient to overwhelm the gravitational effects of the normal matter, are needed.

REFERENCES AND NOTES

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3. M. Visser, *Lorentzian Wormholes—from Einstein to Hawking* (AIP Press, New York, 1995), pp. 115–118.
4. S. W. Hawking and G. F. R. Ellis, *The Large-Scale Structure of Space-Time* (Cambridge Univ. Press, Cambridge, 1973), pp. 263, 266, 271, 272, 292–293, 311, 318, 320, and 354–357.
5. R. M. Wald, *General Relativity* (Univ. of Chicago Press, Chicago, IL, 1984), pp. 226–227, 232, 233, and 237–241.
6. M. Visser, *Lorentzian Wormholes—from Einstein to Hawking* (AIP Press, New York, 1995), pp. 118–119.
7. These classical energy conditions are violated by quantum effects of order $\hbar$, with typical quantum violations being approximately $\langle T_{\mu\nu}\rangle_{\text{violation}} \approx \hbar c^9/(GM)^4$ [M. Visser, *Lorentzian Wormholes—from Einstein to Hawking* (AIP Press, New York, 1995), pp. 128–129]. Here $\hbar$ is Planck's constant, c is the speed of light, G is Newton's constant, and M is the mass of the body under consideration. These quantum effects are not expected to be significant for large classical systems, particularly in cosmological settings. For a general discussion of quantum effects in semiclassical gravity, see N. D. Birrell and P. C. W. Davies, [*Quantum Fields in Curved Spacetime* (Cambridge Univ. Press, Cambridge, 1982)] and S. A. Fulling, [*Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge Univ. Press, Cambridge, 1989)].
8. *General Relativity: An Einstein Centenary Survey*, S. W. Hawking and W. Israel, Eds. (Cambridge Univ. Press, Cambridge, 1979).
9. *Three Hundred Years of Gravitation*, S. W. Hawking and W. Israel, Eds. (Cambridge Univ. Press, Cambridge, 1987).
10. M. Visser, *Lorentzian Wormholes—from Einstein to Hawking* (AIP Press, New York, 1995), pp. 53–73.
11. Look-back time to an object is simply defined as the difference between the age of the universe now and the age of the universe when the light that we are now receiving from that object was emitted.
12. Cosmological inflation is a brief period of anomalously rapid expansion in the early universe during which the universe inflates by an enormous factor. Inflation is commonly invoked as a hypothesis to explain the horizon problem, the flatness problem, and the monopole problem, as discussed in (*13–15*).
13. E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, CA, 1990).
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16. S. Leonard and K. Lake, *Astrophys. J.* **441**, L55 (1995).
17. P. J. E. Peebles, *Principles of Physical Cosmology* (Princeton Univ. Press, Princeton, NJ, 1993).
18. S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), pp. 412–415.
19. A cosmological constant, if present, is absorbed into the definition of the total density and total pressure.
20. Negative energy densities are extremely rare in physics. The only known examples are from small quantum effects [such as the experimentally verified Casimir effect; see M. Visser, *Lorentzian Wormholes—from Einstein to Hawking* (AIP Press, New York, 1995), pp. 121–125] or from a hypothetical negative cosmological constant (see M. Visser, *ibid.*, pp. 129–130). Negative energy does not mean antimatter. Antimatter has positive energy. Negative energy means an energy less than that of the normal undisturbed vacuum.
21. A standard candle is any class of astrophysical objects that is sufficiently well understood, sufficiently well characterized, and has sufficiently nice observational features to be widely accepted by observational astronomers as a useful diagnostic tool. The most famous standard candles are the Cephid variables, whose absolute luminosity is a known function of their period [P. J. Peebles, in (*17*), pp. 20 and 106, and S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), pp. 433–438]. Here I want a similarly well-behaved class of objects in order to trace out galaxy formation.
22. P. J. E. Peebles, in (*17*), p. 106.
23. Here I have expressed the Hubble parameter in terms of the standard astrophysical units of kilometers per second per megaparsec, with a parsec being approximately 3×10^{16} m.
24. P. J. E. Peebles, in (*17*), pp. 610–611. Note the large uncertainties.
25. ———, in (*17*), p. 614. This number is a model-dependent estimate, not an observation. Fortunately, the analysis of this report is relatively insensitive to the precise value of z_f.
26. Particle Data Group, Review of Particle Properties, *Phys. Rev.* **D54** (1996). See the mini-review on pp. 112–114 and references therein. Slightly different numbers are given on p. 66.
27. The standard picture is that the universe is matter-dominated (by dust) out to $z \approx 1000$, so that one expects the equation of state to be $p = 0$. P. J. E. Peebles, in (*17*), p. 100.
28. P. J. E. Peebles, in (*17*), pp. 106–108.
29. S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), pp. 441–451.
30. S. W. Hawking and G. F. R. Ellis, *The Large-Scale Structure of Space-Time* (Cambridge Univ. Press, Cambridge, 1973), p. 95.
31. M. Visser, *Lorentzian Wormholes—from Einstein to Hawking* (AIP Press, New York, 1995), p. 120.
32. Standard variants of inflation are driven by grand unified theory–scale phase transitions in the early universe and take place when energies are of order $kT \approx 10^{14}$ GeV (*13*) with temperatures of order $T \approx 10^{27}$ K, whereas galaxy formation takes place for $T \leq 60$ K.

33. Supported by the U.S. Department of Energy.

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