Prime Formula Weds Number Theory and Quantum Physics

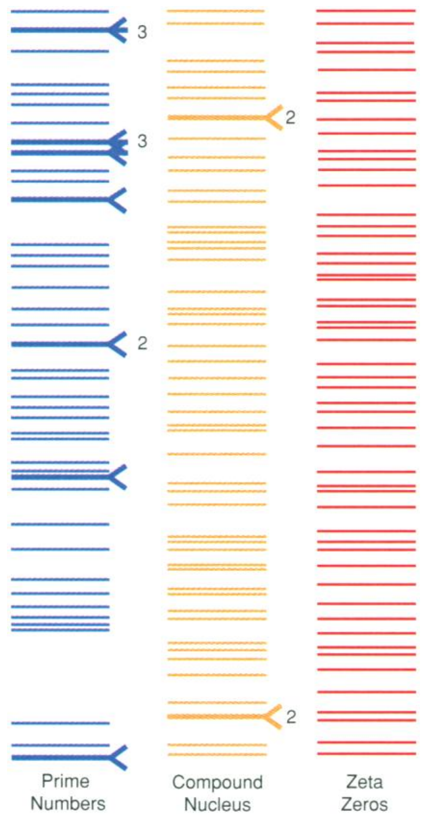
When Pythagoras declared that "All is number," he didn't exactly have in mind the behavior of, say, an excited hydrogen atom in a magnetic field. But two-and-a-half millennia later, an unlikely collaboration of pure mathematicians—the specialists known as analytic number theorists—and quantum physicists is bearing out the philosopher's prescient words. Both groups are hoping that a problem in number theory and a problem in physics will turn out to be two sides of the same numerical coin.

On the one side are the ordinary prime numbers of arithmetic—and a mathematical beast known as the Riemann zeta function, which encodes information about how primes are distributed among the other integers. On the other side is the effort to understand the behavior of complex atomic systems, ranging from the response of a hydrogen atom to a magnetic field to the oscillations of large nuclei. Classical physics suggests that such systems should behave chaotically—but chaos, a hair-trigger instability that makes such systems effectively unpredictable, is anathema to the orderly, linear mathematics of quantum mechanics. Because quantum mechanics provides the "true" description of the atom-scale world, physicists somehow have to reconcile the classical, chaotic description with the quantum-mechanical one. Enter the zeta function.

Theorists on both sides say there are good reasons for believing that the zeta function not only houses information about prime numbers but may also provide a way to simulate quantum chaos on a computer—and thereby test ideas about how to bridge the apparently incompatible chaotic and quantum-mechanical descriptions of the microscopic world. Many are hopeful that the connection with physics will, in turn, break a longtime logjam in pure mathematics by leading to a proof of a century-old problem known as the Riemann hypothesis.

Just why number theory and quantum chaos should be soul mates is a mystery for the gods to unveil. But researchers aren't waiting for the why before exploiting the how. Already, says physicist Michael Berry of the University of Bristol in the United Kingdom, "we have been able to tell mathematicians properties of the Riemann zeta function they didn't know" using ideas from quantum mechanics. "And on the other hand, using mathematics from the Riemann zeta function, we've been able to find new ways of doing calculations in quantum mechanics."

Theorists already had an idea of how to begin reconciling classical chaos with quantum mechanics. They suspected that the wild jumble of trajectories that characterizes classical chaos is reflected in the statistics of the infinite set of energy levels, or "spectrum," of the quantum-mechanical counterpart. More precisely, quantum chaologists such as Berry and Martin Gutzwiller at IBM's Watson Research Center in Yorktown Heights, New York, have proposed that each classical–quantum pair satisfies a "trace formula," a kind of formula that pops up all over mathematics and that distills a complex relationship to its mathematical essence. In the quantum-chaos case, the trace formula relates the lengths of periodic orbits as represented in "phase space"—which keeps track of each particle's changing position and momentum—to combinations of quantum energy levels. Using the trace formula, physicists can convert a system's chaotic behavior in the classical world into predictions about the statistics of its quantum-mechanical spectrum.



Telling resemblance? Energy levels of an excited heavy nucleus are compared with the distribution of prime numbers in the interval 7,791,097 to 7,791,877 and a "spectrum" of Riemann zeta function zeros.

Physicists have more than just theoretical computations convincing them they're on the right track with the trace formula hypothesis. Achim Richter and colleagues at the Technical University of Darmstadt in Germany have done experiments with microwaves in resonators shaped to generate a proliferation of different frequencies—a model for the behavior of electrons in quantum chaos. Other researchers have measured energy levels of actual compound nuclei. However, while the results agree well with certain predictions of the trace formula, the best experiments to date measure at most a few hundred energy levels—too few to really explore the predictions of the theory.

Distilling the essence of chaos. Because it's so hard to test the statistical predictions of the trace formula in actual experiments, quantum chaologists have been looking for ways to simulate the energy levels of these systems on a computer. "The question is, is there a simple model for the quantum mechanics of chaos?" says Berry. The answer may be the Riemann zeta function and its connection with prime numbers.

Mathematicians long ago realized that prime numbers appear with a certain statistical regularity in the sequence of integers. The first to describe this pattern analytically was the German mathematician Bernhard Riemann. In 1859, Riemann sketched an explanation of how the distribution of prime numbers depends on properties of the zeta function, which had been introduced a century earlier by the Swiss mathematician Leonhard Euler.

The zeta function is defined as an infinite series, formed by summing reciprocal powers of the positive integers: $\zeta(s) = 1 + 1/2^s + 1/3^s + 1/4^s + \ldots + 1/n^s + \ldots$. For example, $\zeta(2) = 1 + 1/4 + 1/9 + 1/16 + \ldots + 1/n^2 + \ldots$. Euler had also proved that the zeta function can be written not just as an infinite sum but as an infinite product. Specifically, $\zeta(s) = 1/(1 - 2^{-s})(1 - 3^{-s})(1 - 5^{-s})(1 - 7^{-s}) \ldots (1 - p^{-s}) \ldots$ where each term in the product involves a prime number p. What Riemann saw—and the French mathematician Jacques Hadamard ultimately proved—is that there is a second way to write the zeta function as an infinite product, this time using the "zeros" of the function: values ρ for which $\zeta(\rho) = 0$. The alternative formula is $\zeta(s) = f(s)(1 - s/\rho_1)(1 - s/\rho_2)(1 - s/\rho_3) \ldots (1 - s/\rho_n) \ldots$ where ρ_1, ρ_2, etc., are the zeros of the zeta function and $f(s)$ is a relatively simple fudge factor.

As Riemann observed, the analysis of this formula leads to precise results about the distribution of prime numbers. And to physicists, the equality between the two infinite products looks suspiciously like the trace formula for quantum chaos, with the zeros of the zeta function playing the role of energy levels, and prime numbers (or, more precisely, their logarithms) playing the role of the lengths of periodic orbits.

The first inklings of this connection came in the early 1970s. Hugh Montgomery, a number theorist at the University of Michigan, had discovered a formula that describes the statistics of the spacing between consecutive zeros of the zeta function. During a visit to the Institute for Advanced Study in Princeton, New Jersey, he was introduced to the physicist Freeman Dyson. "Dyson asked me what I was working on, and I told him this result and mentioned this function, and he recognized it as the function that arises in quantum mechanics," Montgomery recalls. To Dyson, the zeros of the zeta function seemed to be behaving precisely like the solutions to the complex mathematical models in statistical mechanics that physicists were then using to calculate energy levels in large nuclei. "It just so happened that he [Dyson] was one of the two or three physicists in the entire world who had worked all of these things out!" Montgomery says.

The next inkling came in the 1980s, when researchers began to grapple with quantum chaos. To theoretical physicists including Oriol Bohigas at the University of Paris, Montgomery's discovery suggested that the zeta function could be used as a stand-in for more complicated models in quantum chaos—and a means of generating truckloads of ersatz energy levels, which could be used to test predictions of the trace formula. What's nice about the zeta function is that its zeros are comparatively easy to compute. With the advent of computers, number theorists have generated zeros by the gigabyte.

Spectral zeros. Over the last decade, Andrew Odlyzko, a mathematician at AT&T Bell Laboratories (now AT&T Labs—Research), has computed hundreds of millions of zeros of the zeta function, often skipping to extremely high levels before beginning the tabulation. (In one series of computations, for example, he identified the 10^{20}th zero of the zeta function, and then calculated the next 500 million zeros.) Odlyzko's data on the spacing between zeros of the zeta function agree almost perfectly with what physicists expect of energy levels in quantum chaos. "It's the first phenomenological insight that the zeros are absolutely, undoubtedly 'spectral' in nature," as the trace formula hypothesis predicts, says Peter Sarnak, a mathematician at Princeton University. The computations are giving physicists access to statistical features they couldn't hope to see in other mathematical models, much less in laboratory experiments.

And if the physicists are right about the zeta function's connection to quantum mechanics, number theorists will be delighted, because that would imply that a conjecture about the location of the function's zeros, known as the Riemann Hypothesis, is also true. The zeros of the zeta function are complex numbers, which means they can be represented as points in the "complex plane"—each has a "real" x coordinate and an "imaginary" y coordinate. They are known to lie within a certain "critical strip," and in his original paper, Riemann made the additional remark that within that strip, the zeros all seem to lie on a straight line. This property, if true, has profound number-theoretic consequences. There is a logjam of results in the mathematics literature that go "If the Riemann Hypothesis is true, then. . . ." Number theorists have been looking for the last hundred years for a bit of dynamite.

The connection with quantum mechanics could provide it. That's because energy levels are necessarily positive numbers, and therefore always lie along a straight line in the complex plane. If researchers could show that the zeta function really does simulate a quantum system, they would be well on their way to proving the Riemann Hypothesis.

No one expects the zeros of the zeta function to pop up in the spectrum of an actual atom, as they are an idealized representation of quantum chaos, but theorists hope they may spot them in an abstract mathematical model of a quantum system. "I'm encouraged that what we're looking for exists," Sarnak concludes. "It doesn't mean we can find it, but what we're looking for is definitely there."

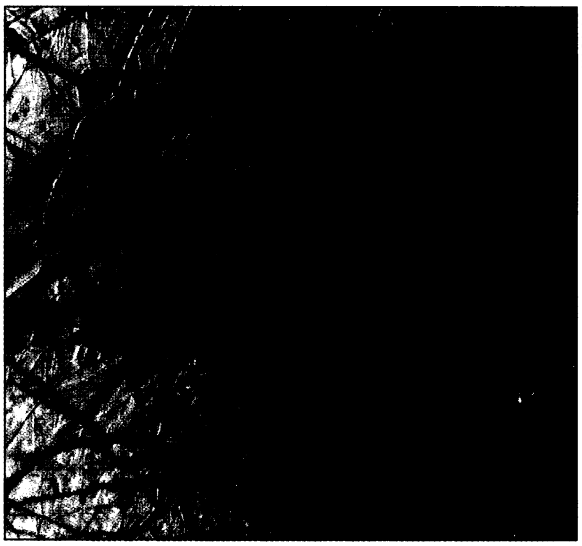
–Barry Cipra

PLANETARY SCIENCE

Does Europa's Ice Hide an Ocean?

An image sent down late last week from the Galileo spacecraft orbiting Jupiter offers the sharpest view yet of the Jovian moon Europa, adding new hints that there may be a water ocean sealed beneath the moon's icy surface—which raises the possibility, however faint, of life on this distant world.

To a geologist's eye, the intricate tangle of crisscrossing ridges seen in the image suggests icy volcanic eruptions. For example, two narrow, squiggly ridges running horizontally through the West Virginia–sized tract of crust look like long fractures where eruptions have spewed icy debris to either side, says Arizona State University planetary geologist Kelly Bender of the Galileo imaging team. Two dark bands angling across the image are reminiscent of Earth's volcanic midocean ridges, with a narrow central ridge bounded on both sides by sets of parallel ridges. That suggests that dirty, heat-softened ice rose to the surface at the central ridge and spread away as new icy crust, says Bender.



NASA/JPL

Europa's furrowed face. Long fractures suggest ice volcanism—and therefore heat or perhaps even liquid water—on this moon.

Where there's volcanism, there's heat. And if there's enough heat, perhaps generated by Jupiter's gentle gravitational massaging of the moon, then there's a good chance for an ocean of liquid water, and therefore life, below the ice-covered surface. And all this ice volcanism must be relatively young, given the dearth of meteorite impact craters; only one 3-kilometer crater (left center) is obvious in this image.

In the coming months, Galileo should return stunning images with 20 times better resolution, more than enough to brighten any geologist's holidays.

–Richard A. Kerr

More Galileo images are available at: http://www.jpl.nasa.gov/releases/ganyhq.html