Long-Range (Casimir) Interactions

Larry Spruch

Normally, nonrelativistic electromagnetic theory with two-particle Coulombic interactions adequately determines the interaction potential of systems A and B if the systems are composed of particles with characteristic velocities much less than the speed of light. If, however, the time it takes light to travel between A and B exceeds a characteristic oscillation period of A or B, the way in which the potential function depends on the separation of the systems can be altered. Called the Casimir effect, it has only recently been confirmed, and it arises in physics, chemistry, and biology. It is the clearest physical manifestation of the fact that, even in a vacuum, electromagnetic fields cannot all vanish.

The development of quantum mechanics in the mid-1920s changed the way we live and remains today at the core of several sciences and technologies. It also changed the way many educated people think about the world: for example, a staple in physics-for-poets courses is a discussion of the profound philosophical implications that one of the elements of quantum mechanics, the uncertainty principle, has had on determinism. Contrast this impact with that of the much-improved 1940s version of quantum mechanics, quantum electrodynamics (QED). With the possible exception of general relativity, QED is arguably the most precise scientific theory ever formulated—theory and experiment agree, in some cases, to some 10 significant figures—and is a cornerstone of modern physics. Yet, much of QED has not been of great practical interest outside physics; many fewer than 10 significant figures are more than adequate for most purposes. Further, QED is more mathematically sophisticated and, hence, generally less simple to interpret than quantum mechanics.

Surprisingly, a subfield of QED initiated by Verwey and Overbeek and by Casimir in the 1940s dealing with systems with large separations, only relatively recently verified experimentally, may turn out to be of enormous importance in many areas. These areas include not only atomic, surface, and condensed matter physics and chemistry, but biology and possibly astrophysics and cosmology. Further, some aspects of this subfield are, at least qualitatively, easily interpretable. The distinction between large and small separations hinges on the finiteness of c, the speed of light (3×10^{10} cm/s). At a sufficiently small separation r, the systems are effectively unchanged during the to-and-fro time of flight, $2r/c$, of photons between the systems; on accounting for the finiteness of c, one therefore introduces only a small relativistic correction. If, however, $2r/c \geq P$, where P is some relevant period of either system, the systems will change substantially during the time required to communicate with one another. The action-at-a-distance Coulomb interaction (which assumes that a charged particle is instantaneously aware of a change in position of any other charge) is then inadequate, and Maxwell's equations, which involve c, must be used; furthermore, the usual velocity-dependent relativistic corrections are not the significant corrections. The effects on the interaction, often referred to as Casimir or retardation effects, can be dramatic (*1–4*).

One interpretation of the difference between QED and ordinary quantum mechanics is that in QED one recognizes that electromagnetic fields fluctuate and can therefore never all be known exactly; in particular, the fields can never all be identically zero (this lack of precision is analogous to the uncertainty principle for particles; one cannot simultaneously know both the position and momentum of a particle). Casimir effects represent the clearest physical manifestation of vacuum fluctuations.

Fluctuations play a much more significant, even dominant, role for atoms or molecules in Rydberg states—where one electron is highly excited and therefore far from all other constituents—than in low-lying states where the constituents are close by. They also play a dominant role in the interaction of distant atoms or electrons with surfaces and of distant surfaces with one another. It is in these areas that high-precision confirmations of retardation effects have been made (*3–5*). Ultimately, Casimir effects may be of much greater importance in biology, where basic systems, such as cells and surfaces, can be large; the interacting constituents may then be far enough apart for retardation effects to be significant (*6*). There also have been hints that truly dramatic Casimir effects may be present in astrophysics and cosmology.

The interaction V between two systems at a separation r, each composed of charged particles with characteristic velocities $v \ll c$, can be roughly sketched as follows. In ordinary quantum mechanics, V is purely Coulombic, whereas in QED, there is an additional contribution to V generated by the exchange of photons. For small r, the Coulomb contribution dominates because the photons contribute only a small v^2/c^2 correction. This can also be true for large r, but there are cases for which, contrary to what one might expect, the c-dependent photon contribution dominates even though $v/c \ll 1$.

The subject of Casimir effects arose in an experimental study of the interaction of molecules in a colloidal suspension (*7*) and stimulated theoretical work. [A number of the examples cited in (*8–22*) will be commented on later.] The atom-atom interaction (*8*) is a special case. The interaction of two metallic walls separated by a vacuum (*9*) is the most famous example. Other interactions include that of two dielectric walls (*10–13*), an atom with a metallic (*14*) or dielectric wall (*10–13*), an electron in a highly excited (Rydberg) state with the ion to which it is bound (*15, 16*), and an electron with a metallic (*17*) or dielectric wall (*18*).

Superluminal group velocities (*19*), the bag model of the nucleon (with the exchange of quarks and gluons rather than photons), and the possible determination of the cosmological constant are other areas of interest. In wetting, the Casimir repulsion in liquid He causes the He to climb the walls of the beaker in which it is contained and determines the thickness as a function of height (*20*). The Casimir repulsion in pentane causes it to spread on water, whereas many other hydrocarbons attract one another and form globules.

The change in the radiative half-life of an excited atom when placed between walls that are close together is an exceedingly interesting effect (*21*). What I will refer to as a "dynamic Casimir effect" can result when acceleration is present (*22*). It has been claimed that Casimir effects enable one to extract energy from the vacuum and to explain the equality of gravitational and inertial masses.

I will give very few experimental details, but some excellent reviews and research papers are available (*3–5*). There is also a wide-ranging theoretical review (*23*).

The author is with the Department of Physics, New York University, 4 Washington Place, New York, NY 10003, USA.

Oddities

There are several curious features of retardation effects. They will be elaborated on later.

1) Relativistic corrections are almost always of relative order v^2/c^2 for $v \ll c$. Consider, however, the interaction $V(r)$ of two hydrogen atoms, each in its ground state, separated by a distance r. For $5a_0 \ll r \ll 137a_0$, where $a_0 \approx 5 \times 10^{-9}$ cm is the Bohr radius, $V(r)$ is the nonrelativistic van der Waals interaction $V_{vdW}(r)$, which falls off as $1/r^6$. As the separation grows to $r \gg 137a_0$, $V(r)$ undergoes a gradual transition to the c-dependent Casimir-Polder interaction $V_{CP}(r)$, which falls off as $1/r^7$ (8). Thus, even though the protons are almost at rest and the electrons have speeds of only $c/137$, the effect is pronounced; the very form of $V(r)$ changes for large r when the finiteness of c is taken into account. Note that V_{CP} is often simpler to interpret than V_{vdW}, even though the theory leading to V_{CP} is much more complicated than the Schroedinger theory that leads to V_{vdW}.

2) Normally, the nonrelativistic approximation to an interaction is obtained from its relativistic version by letting $c \to \infty$. Some retarded interactions, however, are proportional to c, which might appear to imply that the interactions in the nonrelativistic approximation are infinite.

3) The energy density and, consequently, the total energy of an electromagnetic field are both positive in classical theory. If, however, there is a vacuum between two plane-parallel, uncharged conducting plates, the energy density in the region between the plates is negative. (Parenthetically, I remark that domains of negative energy—gravitational energy in particular—may allow the total energy of the universe to be zero. The universe could then have been created from a vacuum without violating conservation of energy.)

4) In a vacuum between two plane-parallel, uncharged conducting plates, light travels perpendicular to the plates at a constant group velocity $v_g > c$. This observation appears to violate a basic premise of the special theory of relativity.

5) No truly relativistic problem is simple enough to allow an exact solution in the context of QED. (There are interesting and informative models, such as that of Jaynes and Cummings, that are solvable.) Instead, one expands the expression for the quantity to be estimated in terms of some smallness parameter γ, which is often proportional to the fine structure constant. Consider the determination of the energy of an atom with atomic number Z in its ground state. The nucleus and the electrons are all close together, so retardation effects are not significant, and the appropriate γ is proportional to Z. For a Rydberg atom, however, retardation effects must be considered, and the appropriate γ decreases as Z increases.

Some Background Material

For a one-dimensional harmonic oscillator of stiffness constant k_{st}, the energy ε_{ho} of a particle of mass m, momentum p_x, and displacement x is $(p_x^2/2m) + (k_{st}x^2/2)$. Classically, ε_{ho} can assume any nonnegative value, but because the uncertainty principle—$\Delta x \Delta p_x \geq \hbar/2$, where Δx and Δp_x are the uncertainties in x and p_x, and $\hbar \approx 6 \times 10^{-27}$ erg·s is Planck's constant divided by 2π—forbids simultaneous exact knowledge of x and p_x, $x = p_x = 0$ is not permitted, nor therefore is $\varepsilon_{ho} = 0$. The allowed values of ε_{ho} are $(n + ½)\hbar\omega$, where $\omega = (k_{st}/m)^{1/2}$ is the classical angular frequency and $n = 0, 1, 2. \ldots$ The minimum energy, $\hbar\omega/2$, is called the "zero-point energy."

The amplitude of any mode of an electromagnetic field behaves like the amplitude of a one-dimensional harmonic oscillator. The energy of a mode of frequency ω can therefore assume any nonnegative value classically, whereas in quantum theory, it is at least $\hbar\omega/2$. Because the number of modes between ω and $\omega + d\omega$ is proportional to $\omega^2 d\omega$, one finds on integration that the total electromagnetic energy is formally infinite, but infinities no longer strike terror in the hearts of physicists; a triumph of QED was learning how to bypass the problems they raise, as in the Casimir effect.

The Casimir Effect

A definition of "Casimir effect" cannot be found in a dictionary, nor is there universal agreement among physicists as to its meaning, but it is often used in connection with retarded interactions between pairs of systems and changes in the energy of the vacuum produced by the imposition of boundary conditions. We begin with a study of the force per unit area F/A between two parallel metallic walls separated by a vacuum, often referred to as *the* Casimir effect. Because the separation z is assumed to be small compared with the wall's dimensions, edge effects can be ignored. The charge e and mass m of an electron play no role in an ideal conductor; the only relevant entities are $\hbar$ and c (quantum theory and relativity theory play a role) and z. On dimensional grounds, F/A is uniquely determined to within a dimensionless constant K and is given by $\bar{F} \equiv F/A = K\hbar c/z^4$ (for arbitrary X, $\bar{X}$ denotes X per unit area). To determine K, consider not two walls but three, with separations z and $L - z$ (Fig. 1). There is energy between the walls because the electromagnetic fields

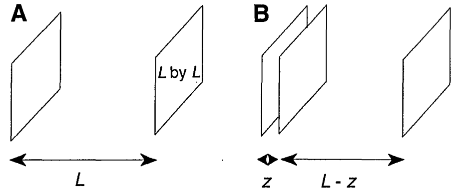

Fig. 1. Despite the apparent simplicity of the two-wall system (**A**), it is simpler to study the Casimir effect for three walls (**B**) because a change in z changes the energy between walls for both cases but changes the energy beyond the walls only for (A).

cannot be identically zero. The allowable modes are those that satisfy the appropriate boundary conditions, including the vanishing of the tangential component of the electric field E at the surface of the wall. The sum of the energies per unit area of the allowable modes is infinite, but the physically relevant energy per unit area between the plates is the difference between the total energy per unit area of the allowable modes in the presence and in the absence of the central wall. For two walls a distance B apart, $\bar{\varepsilon}(B)$ is the energy in a cylinder of unit area and length B, its axis perpendicular to the walls. The physically relevant total energy per unit area between the outer walls is

$$\bar{\varepsilon}_{tot}(L, z) = \bar{\varepsilon}(z) + \bar{\varepsilon}(L-z) - \bar{\varepsilon}(L) \qquad (1)$$

Each of the three terms is infinite because the frequencies of the modes range up to infinity. However, the modes for $\omega \approx \infty$ pass easily through the walls; their contribution is thus independent of the presence of the walls, and the infinities cancel, yielding a finite $\bar{\varepsilon}_{tot}(L, z)$. With $\hbar\omega_n/2$ being the energy of the nth allowable mode of frequency ω_n, one can evaluate $\bar{\varepsilon}_{tot}(L, z)$ for $L \approx \infty$ ($\bar{\varepsilon}_{tot}$ then no longer depends on L) by summing over all modes. One obtains (9) $\bar{\varepsilon}_{tot}(z) = -\pi^2\hbar c/720z^3$. It follows that $\bar{F}$ is attractive and is given by

$$\bar{F} = -\frac{d\bar{\varepsilon}_{tot}(z)}{dz} = -\frac{\pi^2\hbar c}{240z^4} \qquad (2)$$

Even for z as small as 10^{-5} cm, $\bar{F}$ is only about 10^2 dynes/cm^2, or 10^{-4} atm. [Measurements have not been made for walls that can be taken to be ideal. They have been made for dielectric walls, for which $\bar{F}$ is asymptotically proportional to z^{-4} (with a coefficient different from $-\pi^2/240$), and the z^{-4} dependence has there been confirmed (24).]

Why consider three walls when studying the interaction of two? Differentiation with respect to z is achieved conceptually by moving the central wall and keeping the outer walls fixed; the energy beyond the outer walls is thereby unaffected. Had we used only two walls and moved either of them, we would have had to consider the change in energy of the fields in the infinite volume outside the wall that had been moved. A more intuitive derivation of $\overline{F}$ proceeds by considering only two fixed walls and evaluating the net momentum per second per unit area imparted to a wall by the photons between the walls and those outside the wall (25).

Equation 2 leads to the unacceptable result that $\overline{F} \to \infty$ as $c \to \infty$. The difficulty originates in our assumption that the (ideal) conductor responds instantaneously because the to-and-fro time of flight of a photon, $2z/c$, is then always relevant; that is, we are in the retardation zone for any value of z. Now, in fact, any real conductor has some period or characteristic time. Let τ be the longest relevant time interval. The value of τ is irrelevant; the point is that there always exists some nonzero τ. Then $\overline{F}$ assumes the retarded form given by Eq. 2 only for $2z/c \gg \tau$, that is, for $z \gg c\tau \equiv z_0$ ($\overline{F}$ assumes its nonrelativistic form $1/z^3$ for $z \ll z_0$). Thus, z_0 increases with c, and

$$\left|\frac{240}{\pi^2}\overline{F}\right| = \frac{\hbar c}{z^4} \leq \frac{\hbar c}{z_0^4} = \frac{\hbar}{c^3\tau^4} \qquad (3)$$

for $z > c\tau = z_0$. As $c \to \infty$, we have $|\overline{F}| \approx 0$, not $|\overline{F}| \approx \infty$, and the nonretarded form of $\overline{F}$ is valid out to $z = \infty$, as would be expected.

Many theoretical results assume relatively simple forms in asymptotic domains. Nevertheless, the fact that the result for $\overline{F}$ is based formally on a relativistic quantum field theory makes the extraordinary simplicity of the form of $\overline{F}$ in Eq. 2 rather surprising. It is well known that many Casimir effects in the asymptotic domain can be derived starting from the classical theory for electrons, atoms, walls, and the determination of allowable electromagnetic modes; relativistic quantum field theory enters only in the assignment of an energy $\hbar\omega_n/2$ to a classical mode of frequency ω_n (26).

The Casimir-Polder Effect

Consider two ground-state hydrogen atoms, A and B, whose protons are fixed at a separation $r \gg a_0$. If $2r/c \ll P(\approx 10^{-16}$ s), then a Coulomb potential adequately describes the interaction of a component of A with one of B (quantization of the electromagnetic field produces only a small correction). Loosely speaking, A responds at any instant to the orientation of the dipole moment of B at that instant, and vice versa; the two dipoles orient themselves in the state of minimum energy and rotate locked in that state (Fig. 2A). If $2r/c \gg P$, however, a Coulombic description is inadequate. Thus, a photon emitted by B carrying information about the orientation of its dipole reaches A a time r/c later, by which time the orientation of B's dipole will have changed significantly, and the dipole of A will adjust to the dipole of B at the earlier time (Fig. 2B). To account for the finiteness of c and the associated time delay, one must use Maxwell's equations. Because the atoms are in their ground states, any photons emitted must be (quantized) virtual photons, with an energy for frequency ω of $\hbar\omega/2$, and the very nature of the interaction is changed. Thus, when it is necessary to account for retardation, which would seem to be a classical effect, it becomes necessary to quantize the electromagnetic fields [for $r \gg a_0$, the fractional change in the distance between the electron (e^-) or proton (p) in A and the e^- or p in B is small as the e^-'s move in their orbits; for $2r/c \gg P$, it is the fact that the dipoles are oriented differently than for $2r/c \ll P$ that changes the $1/r^6$ to $1/r^7$].

Dielectric Walls

The discussion leading to Eq. 2 of $\overline{F}$ for metallic walls omitted the (not terribly difficult) calculational details. The details are quite complicated for $\overline{F}_{DD}$, the force per unit area between uniform dielectric walls. [I will henceforth use the subscripts M, D, Dil, At, and El to denote a metallic (ideal) wall, a dielectric wall, a wall composed of a dilute gas of atoms, an atom, and an electron, respectively, and X denotes any one of these. Accordingly, $\overline{F}$ of Eq. 2 will be written as $\overline{F}_{MM}$.]

The complexity is caused in part by the fact that a dielectric wall is characterized by a frequency-dependent dielectric "constant," the permittivity $\varepsilon(\omega)$. For metallic walls, there is only a retardation zone, but for dielectric walls, there are in addition unretarded and transition zones. We will restrict our attention to large separations, that is, to the retardation zone. Only low-frequency vacuum fluctuation fields then contribute significantly. The integrals that define V_{XD} contain the factor e^{ikr}, where r is a wall-wall, atom-wall, or electron-wall separation; large values of ω, those for which $k = \omega/c$ is greater than $1/r$, therefore tend to average out. Values of ω much smaller than c/r are also insignificant, because the number of modes between ω and $\omega + d\omega$ is proportional to $\omega^2 d\omega$. The dominant contribution comes from the neighborhood of $k = 1/r$, or $\omega = c/r$, which is arbitrarily small for arbitrarily large r. We can therefore approximate $\varepsilon(\omega)$ by $\varepsilon(0) \equiv \varepsilon_0$, a great simplification.

Expressions for $\overline{F}_{DD}$ and V_{AtD} (3, 10, 11) and for V_{ElD} (18) are known, but they are so complicated that approximations, if simple and reasonably accurate, can be very useful. Rather than considering approximations to V_{XD}, for heuristic purposes we obtain an approximation to the much simpler textbook problem of the nonretarded e^--wall interaction, $v_{ElD}(z, \varepsilon_0)$, where the electron, in a vacuum, is at a distance z from the wall (27), and we use v_{ElD} rather than V_{ElD}, to stress its nonretarded nature. Because this is a static problem, we set $\omega = 0$ and $\varepsilon(\omega) = \varepsilon_0$. The origin of v_{ElD} is very different from the origin of V_{XD}, vacuum fluctuations playing no role, but the methods of approximation are almost identical in form.

Because v_{ElD} must vanish for $\varepsilon_0 = 1$ (the wall is then a vacuum), we choose v_{ElD} to be proportional to $\varepsilon_0 - 1$. For $\varepsilon_0 = \infty$, the wall is a metal, and the image method gives $v_{ElM}(z) = -e^2/(4z)$. This suggests that we choose

$$v_{ElD}(z, \varepsilon_0) = -\frac{\varepsilon_0 - 1}{\varepsilon_0 + b}\frac{e^2}{4z} \qquad (4)$$

where b is a dimensionless constant independent of ε_0. To fix b, we take the wall to be dilute and obtain two different expressions for v_{ElDil}. On the one hand, v_{ElDil} is a superposition of nonretarded electron-atom interactions, given by

$$v_{ElAt}(r) = -\frac{1}{2}\frac{\alpha_0 e^2}{r^4} \qquad (5)$$

The interaction of an electric dipole moment μ with the static electric field $\mathbf{E}$ that induces it is $-\mu\cdot\mathbf{E}/2$, where $\mu = \alpha_0\mathbf{E}$ and α_0 is the static electric dipole polarizability of the atom. The interaction is then $-\alpha_0 E^2/2$. Because the field seen by the atom is e/r^2, we obtain Eq. 5. Integrating over the full domain of the dilute wall, we find

$$v_{ElDil}(z, \varepsilon_0) = -\frac{\pi N_{At}\alpha_0 e^2}{2z} \qquad (6)$$

where N_{At} is the number density of atoms. We now use the standard relation (for a dilute gas) connecting the macroscopic or bulk property ε_0 with the microscopic or

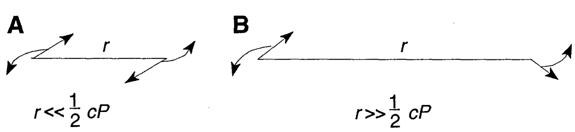

Fig. 2. Schematic plots in a study of an atom-atom interaction for $r \gg a_0$ (a_0 is an atomic size) for **(A)** $2r/c \ll P$ and **(B)** $2r/c \gg$ P. In (A), the dipoles interact effectively instantaneously, rotate in lock-step in the state of minimum energy at all times, and generate the van der Waals $1/r^6$ interaction. In (B), each dipole responds to the dipole moment of the other from an earlier time, they are each in a different minimum-energy state, and they generate the weakened Casimir-Polder $1/r^7$ interaction.

atomic property α_0, namely, $\varepsilon_0 - 1 = 4\pi N_{At}\alpha_0$ (28). (For a dilute gas, $\varepsilon_0 - 1$ must be proportional to N_{At} and to some property of an atom, and the polarizability α_0 is the property to be expected.) Equation 5 can then be rewritten as

$$v_{\rm EIDil}(z, \varepsilon_0) = -\frac{(\varepsilon_0 - 1)e^2}{8z} \quad (7)$$

Alternatively, $v_{\rm EID}$ follows from Eq. 4 upon replacement of ε_0 in the denominator (but not in the numerator) by unity. Thus, comparing Eq. 4 with $\varepsilon_0 + b$ replaced by $1 + b$ and Eq. 7, we find $b = 1$, which gives

$$v_{\rm EID}(z, \varepsilon_0) = -\frac{\varepsilon_0 - 1}{\varepsilon_0 + 1}\frac{e^2}{4z} \quad (8)$$

as an approximation for arbitrary ε_0 (obtained by demanding that it be exact for $\varepsilon_0 = \infty$ and become exact as $\varepsilon_0 \sim 1$). This particular example is a bit misleading, for it happens to give the exact result. If the process is reversed, we can obtain $v_{\rm EIAt}$ from knowledge of $v_{\rm EID}$.

Similarly, we can obtain approximations for $V_{\rm DD}$ and $V_{\rm EID}$ from known results on $V_{\rm AtM}$, $V_{\rm AtEI}$, and $V_{\rm EIM}$, and with a slight modification, we can approximate $\overline{F}_{\rm DD}$. The approximations are accurate to within about 15% over the range $1 \leq \varepsilon_0 \leq \infty$. It was by no means certain that the approach that worked so well for the simple classical interaction $v_{\rm EID}$, an approach without an honest basis, would work at all well for the complicated quantum interactions $V_{\rm XD}$ and $\overline{F}_{\rm DD}$, but the good results obtained are not really surprising; $V_{\rm XD}$ and $\overline{F}_{\rm DD}$ are approximated by using quantum results for simpler but related situations, and at very large separations we can, as noted, replace $\varepsilon(\omega)$ by ε_0, thereby eliminating much of the structural complexity of the problem.

He_2, Dynamic Casimir Effects, and Superluminal Velocities

There is a multitude of known and possible Casimir effects (23, 29); understanding in some cases is far from complete.

"*The weakest bond.*" Whether two helium atoms, with their small atomic polarizabilities and small masses, could form a bound state (He_2) has fascinated many chemists and physicists since the 1920s. It followed from an analysis of scattering data of virial coefficients that He_2, if it existed, had a binding energy E_β of a few millikelvin (that is, $\sim 10^{-7}$ eV). Some beautiful, recent experiments have shown that He_2 does exist. Its wave function decays asymptotically as e^{-Kr}, where $1/K = (\hbar^2/M_\alpha E_\beta)^{1/2} \approx 10^{-6}$ cm, with M_α being the mass of the helium nucleus. Although the two atoms will most often be found at $r \ll 10^{-6}$ cm, the non-negligible probability of finding them at $r \cong 10^{-6}$ cm leads to remarkably large retardation effects of about 10% on the binding energy and about 5% on the mean separation (30).

Dynamic Casimir effects. Accelerating conducting walls can convert virtual photons to real photons, but the effects are significant only for accelerations that are so large as to be unobtainable by mechanical means. However, one can simulate a rapidly accelerating conducting wall by causing a dielectric wall to become conducting in femtoseconds (22).

Superluminal velocities. The group velocity v_g of a pulse, the velocity of the peak, is normally physically meaningful. In QED, for light traveling between and perpendicular to a pair of plates, v_g is constant and exceeds c, the speed of light in free space (19). The special theory of relativity demands that the signal velocity $c_{\rm sig}$, the velocity at which information is transmitted, not exceed c. The uncertainty principle makes it impossible to know the exact times of emission and detection of a pulse, and one finds that $c_{\rm sig}$ does not exceed c (31) ($c_{\rm sig} > c$ would allow many science fiction–type delights).

As conceptually fascinating as Casimir effects may be, confirmations have just begun, and significant applications have yet to appear. However, research on the subject represents a growth industry, and there is reason to expect that not only will significant effects be found, but that they will be found in a variety of domains.

REFERENCES AND NOTES

1. Reviews that emphasize the topics covered in this article include L. Spruch, *Phys. Today* **39**, 37 (November 1986), and (2–4).
2. F. S. Levin and D. A. Micha, Eds., *Long-Range Casimir Forces: Theory and Experiment in Multiparticle Dynamics* (Plenum, New York, 1993).
3. P. W. Milonni, *The Quantum Vacuum* (Academic, New York, 1993), especially chaps. 7 and 8.
4. E. Hinds, in *Cavity Quantum Electrodynamics*, suppl. to *Advances in Atomic, Molecular, and Optical Physics*, P. R. Berman, Ed. (Academic Press, New York, 1994), chap. 1.
5. S. R. Lundeen, in (2), chap. 2.
6. See, for example, S. Nir, *Prog. Surf. Sci.* **8**, 1 (1976).
7. E. J. W. Verwey and J. T. G. Overbeek, *Theory of the Stability of Lyophobic Colloids* (Elsevier, Amsterdam, 1948). See also M. J. Sparnaay, in *Physics in the Making*, A. Sarlemijn and M. J. Sparnaay, Eds. (Elsevier, Amsterdam, 1989), pp. 235–246; B. de Witt, *ibid.*, pp. 247–272.
8. H. B. G. Casimir and D. Polder, *Phys. Rev.* **73**, 360 (1948).
9. H. B. G. Casimir, *Proc. K. Ned. Akad. Wet.* **60**, 793 (1948).
10. E. M. Lifshitz, *J. Exp. Theor. Phys. USSR* **29**, 94 (1955); L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, vol. 2 of *Course of Theoretical Physics* (Pergamon, New York, 4th revised English ed., 1975).
11. I. E. Dzyaloshinskii, E. M. Lifshitz, L. Pitaevskii, *Adv. Phys.* **10**, 165 (1961).
12. N. G. van Kampen and co-workers greatly simplified the analysis. See (3).
13. For $V(r)$ for two sets of walls, see F. Zhou and L. Spruch, *Phys. Rev. A* **52**, 297 (1995).
14. H. B. G. Casimir, *J. Chim. Phys.* **46**, 407 (1949).
15. E. J. Kelsey and L. Spruch, *Phys. Rev. A* **18**, 15 (1978). Only $V(r)$ for $r \sim \infty$ was obtained. For $V(r)$ for all r, see (16); see also J. Bernabéu and R. Tarrach, *Ann. Phys. NY* **102**, 323 (1976).
16. G. Feinberg and J. Sucher, *Phys. Rev. A* **27**, 1958 (1983); C. K. Au, G. Feinberg, J. Sucher, *Phys. Rev. Lett.* **53**, 1145 (1984); G. Feinberg, J. Sucher, C. K. Au, *Phys. Rep.* **180**, 83 (1989). See also J. F. Babb and L. Spruch, *Phys. Rev. A* **36**, 656 (1987).
17. G. Barton, *J. Phys. A* **10**, 601 (1977). See also L. Spruch and E. J. Kelsey, *Phys. Rev. A* **18**, 845 (1978); R. Shakeshaft and L. Spruch, *ibid.* **22**, 811 (1980).
18. Y. Tikochinsky and L. Spruch, *Phys. Rev. A* **48**, 4223 (1993).
19. K. Scharnhorst, *Phys. Lett. B* **236**, 354 (1990); G. Barton, *ibid.* **237**, 559 (1990).
20. Dzyaloshinskii *et al.* (11) discuss some of the properties of liquid helium.
21. As an extreme case, consider an atom in an excited state, which, in free space, would radiate a photon of frequency ν. If placed in a wave guide whose cutoff frequency is higher than ν, then as pointed out by D. Kleppner, the atom could not radiate. See (4), sect. IVA; G. Raithel *et al.*, in (4), chap. II, sect. IIA; and references therein.
22. E. Yablonovitch, in *Resonances. A Volume in Honor of the 70th Birthday of N. Bloembergen*, M. D. Levenson, E. Mazur, P. S. Pershan, Y. R. Shen, Eds. (World Scientific, Rivers Edge, NJ, 1990), p. 112.
23. For a review that is rigorous and casts a much wider net than does the present article, with coverage of elementary particles (including the bag model) and cosmology, considerable material on topological effects, and many solved model problems, see V. M. Mostepanenko and N. N. T. Trunov, *Sov. Phys. Usp.* **31**, 965 (1988). [transl: *Usp. Fiz. Nauk* **156**, 385 (1988)].
24. B. V. Derjaguin and I. I. Abrikosova, *Sov. Phys. JETP* **3**, 819 (1957); M. J. Sparnaay, *Physica* **24**, 751 (1958); J. N. Israelachvili and D. Tabor, *Proc. R. Soc. London Ser. A* **331**, 19 (1972).
25. P. W. Milonni, R. J. Cook, M. E. Goggin, *Phys. Rev. A* **38**, 1621 (1988).
26. See, for example, E. A. Power, *Introductory Quantum Electrodynamics* (Elsevier, Amsterdam, 1964). See also L. Spruch, J. F. Babb, F. Zhou, *Phys. Rev. A* **49**, 2476 (1994).
27. L. Spruch and Y. Tikochinsky, *Phys. Rev. A* **48**, 4213 (1993).
28. We could use the Clausius-Mosotti relation, as has been done to good effect by P. W. Milonni and M.-L. Shih [in *Contemporary Physics*, J. S. Douglas, Ed. (Taylor and Francis, Bristol, PA, 1993), vol. 33, p. 313], but we need the relation only for $\varepsilon - 1 \sim 0$. The linear relation is simpler, and the results are comparable.
29. For a somewhat different but interesting approach, see V. F. Ginzburg, *Theoretical Physics and Astrophysics* (Pergamon, Oxford, 1979), chap. 13.
30. F. Luo, G. C. McBane, G. Kim, C. F. Giese, W. R. Gemtry, *J. Chem. Phys.* **98**, 3564 (1993); *ibid.* **100**, 4023 (1994); W. Schöllkopf and J. P. Toennies, *Science* **266**, 1345 (1994); F. Luo, C. F. Giese, W. R. Gentry, *J. Chem. Phys.* **104**, 1151 (1996); see also M. J. Jamieson *et al.*, *Phys. Rev. A* **51**, 2626 (1995); A. R. Janzen and R. A. Aziz, *J. Chem. Phys.* **103**, 9626 (1995); H. L. Williams *et al.*, unpublished material.
31. P. W. Milonni and K. Svozil, *Phys. Lett. B.* **248**, 437 (1990); S. Ben-Menahim, *ibid.* **250**, 133 (1990).

32. Supported in part by the National Science Foundation under grant PHY-9400673 and by the Humboldt Foundation. Part of the research was done at the Max-Planck-Institut für Kernphysik, Heidelberg, and at the Physikalisches Institut der Universität Heidelberg.