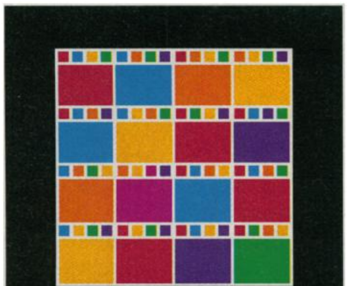
MATHEMATICS

New Proof Makes Light Work Of Partial Latin Squares

Mathematicians are fond of ending proofs with the letters "QED." The abbreviation officially stands for "Quod Erat Demonstrandum"—"which was to be proved"—but mathematical wags prefer a snappier English version: Quite Easily Done.

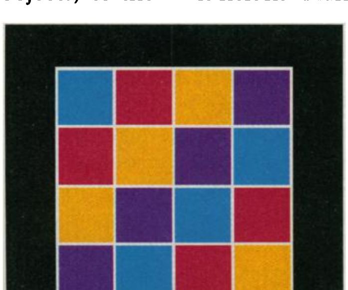


A PARTIAL LATIN SQUARE

Sometimes it really does look that way. A case in point is a recent solution to a stumper known as the Dinitz conjecture. First posed in the late 1970s, this challenge in combinatorics—the field of mathematics concerned with ways of counting and arranging the members of finite sets—eluded attempts to solve it for more than 15 years and earned a reputation for difficulty. But in 1992, Jeannette Janssen, a graduate student at Lehigh University in Bethlehem, Pennsylvania, startled experts by solving a close variant of the problem (*Science*, 5 March 1993, p. 1404). Now, inspired by Janssen's approach, Fred Galvin at the University of Kansas has polished off the original problem with an ease that surprised even Galvin himself.

"Things move pretty quickly in this business sometimes," observes Herb Wilf, a combinatorialist at the University of Pennsylvania, who had predicted a year ago that Janssen's result would "galvanize" work on the subject. Galvin's solution, which he wrote up as a letter that was circulated to his colleagues (a tradition in certain parts of mathematics), has theorists buzzing with the possibility that other, related problems may also be easier to solve than they now appear. "One of the nicest things about mathematics is that you can never tell" how difficult a problem really is, says combinatorialist Jeff Kahn of Rutgers University.

The Dinitz problem might have appealed to the artist Paul Klee. The basic challenge is to assign colors (or objects) to the spaces of an $n \times n$ grid in such a way that no color is used twice in any row or column. If the same n colors are available for each space, it's easy to figure out a rule for doing so (see figure at right). Such arrangements, called Latin squares, have been studied by mathematicians for centuries. But in 1977, Jeff Dinitz, then a graduate student at the Ohio State University (now at the University of Vermont) dreamed up a variant, called partial Latin squares, that no one had considered before.

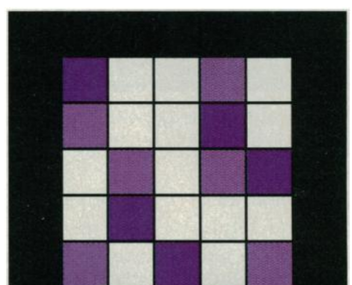


A LATIN SQUARE

ILLUSTRATIONS BY C. FABER SMITH

Suppose that, instead of being restricted to the same set of n colors, each space has its own set of n available colors, which can overlap to various degrees (see figure at left). The total number of different colors is greater than n, because the sets can be different. Dinitz conjectured that no matter which sets of colors are assigned to the spaces, it will always be possible to color each square without choosing at least one color twice in some row or column. Although it may seem intuitively obvious that if the problem can be solved for n colors, it should be solvable for a number greater than n, there's no obvious rule to follow—and thus no assurance that it can always be solved.

The problem is an instance of a more general problem in graph theory: coloring the edges of a graph (that is, lines connecting pairs of vertices) so that no two edges of the same color meet at a vertex. Such problems sound playful, but they underlie many applications in computer science, such as efficient scheduling and network routing. For example, each edge may represent a "conversation" between two computer processors, and each color may represent a specific time; the constraint is that a processor can't carry on two conversations at once.



A KERNEL

Dinitz's conjecture—popularized by the itinerant Hungarian mathematician Paul Erdös—quickly gained a reputation as being very difficult to prove. And that makes the simplicity of Galvin's solution all the more startling to his colleagues. "The proof is just amazing," says Kahn. What's more, according to Galvin, none of the ideas in the proof is new: "All I did was put together a couple of things that were already in the literature."

Galvin's starting point was Janssen's result: a proof that the coloring problem can always be solved if you have a set of $n + 1$ choices for each space in a partial Latin square. Janssen had built on work by Noga Alon and Michael Tarsi at Tel Aviv University, who had introduced new algebraic techniques to attack the problem. When Galvin referred to their paper, he found a remark that led him to the crucial realization: If he could prove that certain graphs associated with the Dinitz problem have a technical feature called kernels—subsets of vertices corresponding to a collection of identically colored spaces, no two of which are in the same row or column—he could solve the entire problem.

Not being an expert on kernels, Galvin went to the library. In the second paper he looked at, by Frédéric Maffray at the University of Toronto, Galvin found exactly the result he needed. Maffray's work guaranteed the existence of kernels for a large class of graphs, including those associated with the Dinitz problem. "I felt a little like a thief—like I'd found a check lying in the street and took it to the bank and cashed it," Galvin jokes.

What Galvin got when he cashed in this earlier work was, in effect, an algorithm for choosing colors. The first step in the algorithm picks a color, say purple, and locates all spaces whose list of available colors includes purple. The next, key step is to identify a kernel for the set of potentially purple spaces—and then color all the cells in the kernel purple (dark spaces in figure at left). The algorithm then repeats this process with the remaining spaces and the remaining colors. In essence, Maffray's theorem says you never run into a color that lacks a kernel, while the remark in the Alon-Tarsi paper implies that you run out of spaces in the partial Latin square before you run out of colors.

Original or not, researchers are delighted with Galvin's proof, which takes up less than a page. "This is an example of a proof out of 'The Book,'" says Ron Graham, adjunct director in the information sciences division at AT&T Bell Labs. "The Book" is a hypothetical compilation of beautiful proofs, which Erdös postulates resides with God (whom Erdös refers to as the "Supreme Fascist"). Galvin, one of Erdös's many protégés, is modest. The proof is simple enough to be understood by high school students, he says—"and maybe even a bright high school student could have thought up the whole proof."

–Barry Cipra