Dynamics of Elliptical Galaxies

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Elliptical galaxies were once thought to be similar in their structure and dynamics to rotationally flattened bodies like stars. The discovery that elliptical galaxies rotate much more slowly than a fluid body with the same shape has led to a qualitative change in our understanding of the dynamics of these systems. It is now believed that elliptical galaxies are fully triaxial in shape. Self-consistent triaxial equilibria have been constructed and appear to be long-lived; they are made possible by the existence of conserved quantities, or integrals of motion, for galactic potentials without rotational symmetry. Many self-consistent equilibria are unstable; the nonexistence of elliptical galaxies with axis ratios more extreme than 3:1 is probably the result of such an instability. There is evidence for strong central mass concentrations, perhaps massive black holes, at the centers of some nearby galaxies. Recent observations suggest that many elliptical galaxies formed through the merger of two or more spiral galaxies.

About one-third of the bright galaxies in the universe fall into the class of elliptical galaxies. In contrast to spiral galaxies, like our own Milky Way, which contain a prominent disk composed of young stars, gas, and dust, elliptical galaxies have a smooth, almost featureless appearance (Fig. 1). The only obvious characteristic that distinguishes one elliptical galaxy from another—aside from size or total luminosity, quantities that depend on the assumed distance—is apparent axis ratio. This fact is reflected in the classification scheme that was proposed by Hubble in 1936 and is still widely used (*1*). Although spiral galaxies are classified according to the detailed morphology of their spiral arms, elliptical galaxies are distinguished only by their apparent elongation: an elliptical galaxy with axis ratio b/a, where a is the length of the semimajor axis and b is the length of the semiminor axis, has a Hubble type of En, where $n = 10(1 - b/a)$. Thus, an apparently round galaxy has Hubble type E0, and the most elongated ellipticals are of type E7 (*2*). Because we see only the stellar distribution that is projected along our line of sight, it is impossible to directly determine the intrinsic elongation of any single elliptical galaxy or indeed to decide whether elliptical galaxies as a class are axisymmetric (oblate or prolate) or fully ellipsoidal ("triaxial" in the nomenclature of astronomers) (*3*).

Until about 1975, it was commonly (though in retrospect, rather loosely) assumed that elliptical galaxies were rotationally flattened and oblate, the stellar-dynamical counterparts to Maclaurin spheroids (*4*). Thus, elliptical galaxies were seen as logical complements to the more rapidly rotating, and hence more strongly flattened, spirals. However, the first accurate measurements of the rotation velocities of elliptical galaxies revealed a significantly lower rotation rate than expected for an equivalent fluid body, even one with the density structure of a real galaxy (*5*). Elliptical galaxies were revealed to be "hot" stellar systems, in which most of the support against gravitational collapse comes from essentially random motions, rather than "cold" systems, like spiral galaxies, in which ordered rotation contributes most of the internal kinetic energy. Two questions immediately arose from these observations: What produces the observed flattening; and, given that rotation plays only a minor role, are elliptical galaxies axisymmetric or fully triaxial (*6*)? Finding the answers to these questions, and others that have arisen as the quality of the observational data has increased, has occupied the majority of galactic dynamicists for the last decade or so. This article reviews recent progress in our understanding of the structure and dynamics of elliptical galaxies.

Stellar systems differ in fundamental ways from other large-N systems, such as gases or plasmas. The molecules in a gas interact by means of short-range forces, which quickly act to randomize the distribution of molecular velocities at each point. The stars in a galaxy rarely undergo physical collisions, except perhaps in regions of atypically high density, such as galactic nuclei. Furthermore, it is easy to show that, because of the long-range nature of the gravitational force, the perturbing effect of gravitational encounters with nearby stars is typically negligible compared with the overall gravitational force of the galaxy. As a result, the gravitational force on any star will not vary rapidly, and each star moves smoothly through the force field produced by the galaxy as a whole. More quantitatively, the relaxation time, that is, the time scale over which gravitational encounters with individual stars produce significant changes in a star's orbit, is long compared to the crossing time, the time required for a star to move from one side of the galaxy to the other. For a typical elliptical galaxy, the mean relaxation time is of order 10^{15} years, whereas the crossing time is about 10^{9} years. Astronomers therefore refer to galaxies as collisionless systems.

The particles in a fully ionized plasma also interact by means of long-range forces, and low-density plasmas share with stellar systems the property that the mean field is more important than the fields of nearby particles. However, plasmas contain both positive and negative charges, so they are neutral on large scales (*7*). Because gravity is always an attractive force, stellar systems tend to form strongly inhomogeneous equilibria in which the gravitational force is balanced at every point by the stellar "pressure." This simple fact greatly complicates the theoretical study of galaxies compared with that of plasmas because even a zeroth-order description of a galaxy typically requires an inhomogeneous model (*8*). The problem is particularly severe for hot stellar

Fig. 1. A large elliptical galaxy, M87, in the core of the Virgo galaxy cluster. The center of the image is burned out through overexposure, whereas the envelope is underexposed and almost invisible. The bright knots on one side are produced by a jet of hot gas that is being ejected from the nucleus, perhaps by a massive central black hole. This is one of the few elliptical galaxies with obvious substructure. [Photo by A. R. Sandage with the 200-inch Hale telescope; courtesy of J. Bedke at the Space Telescope Science Institute]

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systems like elliptical galaxies, in which the stars pass through regions of very different gravitational force in one crossing time. The collective gravitational effects that determine the structure and evolution of a hot stellar system are therefore usually too complex to be described by simple mathematical models, and most studies of these systems must be based on computer simulations. Recent advances in the art of studying the gravitational *N*-body problem have contributed substantially to progress in this field (9).

Orbits and Shapes

The dynamics of stellar systems are closely tied to the forms of the trajectories, or orbits, followed by individual stars. The classical theory of orbits, dating back to Kepler and Newton, is based on the inverse-square force law that describes two interacting point masses. In a galaxy, the force field derives from all the stars (as well as nonluminous components, such as dust and dark matter), and the dependence of the gravitational acceleration on distance from the galactic center is rarely close to inverse-square. The orbits are therefore very different from the familiar closed ellipses of the two-body problem.

In an arbitrary spherical potential, the orbital motion can be well approximated as a precessing ellipse that fills a plane, annular region. In axisymmetric potentials, these annular orbits precess around the symmetry axis, producing time-averaged orbital densities that are roughly doughnut-shaped. These shapes can be discovered by direct integration of trajectories on a computer, but they can also be predicted through study of the mathematical form of the integrals of motion associated with realistic galactic potentials.

An integral of motion is any function of the phase-space coordinates (**x**, **v**) that is constant along orbits. Examples are the so-called "classical" integrals: the total energy, which is always an integral of motion if the gravitational field is not time-varying, and the angular momentum, which is conserved about any axis of rotational symmetry. Because integrals of motion are relations between the phase-space variables, they restrict the volume in which a star can move; for instance, an orbit respecting only the energy integral would fill a volume defined by an equipotential surface. Numerical integration reveals that the equations of motion in many three-dimensional potentials seem to respect integrals beyond the classical ones: the orbits confine themselves to regions much smaller than the regions defined by the classical integrals (*10*).

Unfortunately, the class of three-dimensional mass models for which the nonclassical integrals of motion can be expressed in closed form, or can even be shown mathematically to exist, is very small. The mathematician Stäckel first classified, in 1890, the full set of potentials for which the Hamilton-Jacobi equation becomes separable in curvilinear coordinates to yield three exact integrals of motion (*11*). Remarkably, it required a further 80 years for the realization that certain ellipsoidally stratified mass models have potentials of the Stäckel form (*12*). The most general such mass distribution is

$$\rho = \rho(m^2) = \frac{\rho_0}{(1 + m^2)^2}$$

$$m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \qquad (1)$$

The mass density, ρ, is constant on surfaces of constant m, which are ellipsoids. These ellipsoids can have arbitrary axis ratios.

The orbits in the force field generated by the mass model of Eq. 1 fall into four families (Fig. 2). Three of these families are "tubes," orbits that respect an integral similar to the angular momentum around one of the three axes, which causes them to avoid the center. Orbits of the fourth family are called "boxes," which are unique to triaxial potentials. At low energies, and thus small amplitudes, the box orbits are rectangular parallelepipeds confined to the core, where the equations of motion are those of a three-dimensional, incommensurate harmonic oscillator. At large energies, the box orbits become elongated in the direction of the long axis of the ellipsoid. Box orbits are distinguished from tube orbits in that they pass arbitrarily close to the center and have an alternating sense of rotation. Numerical integrations show that—with some important exceptions discussed below—these families of orbits are the main ones in a fairly wide class of triaxial potentials (*13*).

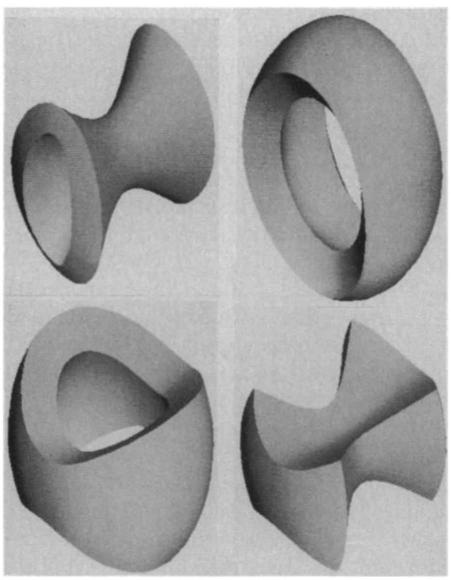

Fig. 2. The four families of orbits in an integrable, nonrotating, triaxial potential. The pictures are perspective views of solid bodies that envelope the volumes filled by the four types of orbits. The symmetry axes of the solids coincide with the symmetry axes of the galaxy. [Reprinted from (*14*) with permission © American Astronomical Society]

A first step in determining whether real elliptical galaxies are triaxial (as opposed to axisymmetric) is to show that self-consistent triaxial equilibria exist. One simple, although approximate, way to accomplish this is by numerically combining a discrete set of orbits like those in Fig. 2 in such a way as to reproduce the mass distribution of Eq. 1. Such experiments suggest that triaxial equilibria do indeed exist for essentially any choice of axis ratios (*13, 14*). These models typically contain sizable contributions from each of the four basic families; the box orbits, which are elongated in the same direction as the galaxy and are concentrated near the center, are often the dominant component. An alternative way to verify the existence of triaxial equilibria is to use an *N*-body program to simulate the relaxation of a set of stars from nonequilibrium initial conditions, such as an extended protogalactic cloud or a pair of subsystems in the process of merging. One finds that the final state is often substantially triaxial and that many of the stellar orbits have the same basic properties as the orbits in integrable potentials (*15*). Taken together, these two sorts of experiments suggest that triaxial ellipsoids are natural, and perhaps long-lived, shapes for slowly rotating galaxies (*16*).

Real elliptical galaxies have density profiles that are much more centrally condensed than that of the integrable model described by Eq. 1. High-resolution observations of nearby ellipticals and the central bulges of spiral galaxies, including the Milky Way, often reveal core luminosity profiles that rise very steeply into the center with no discernible limiting density (*17*). The bulge of the nearby spiral galaxy M31 contains a dense nucleus of almost stellar appearance (*18*), and many elliptical galaxies may contain massive black holes at their centers, as discussed below. The gravitational potentials of triaxial models with density profiles similar to these are far from Stäckel-like, and, although many of the orbits in such model potentials appear to respect three integrals of motion, the orbital shapes are often quite different from those in the analytically separable potentials.

Most strongly affected are the box orbits, which pass arbitrarily close to the center after sufficient time. Numerical orbit integrations show that most box orbits are destroyed when the stellar density increases more rapidly than r^{-1} near the center, as appears to be the case for many resolved galaxies. Some of these box orbits become "stochastic," or chaotic, filling volumes that are roughly bounded by equipotential surfaces. Other box orbits are replaced by so-called "boxlets," orbits similar to boxes but which curve in such a way as to avoid the center (19). It is unknown at present whether self-consistent triaxial models with central cusps can be constructed from these orbits. If the answer is no, this would imply that many elliptical galaxies are axisymmetric, or perhaps that their triaxial figures are slowly evolving toward axisymmetry.

An interesting problem that has not been resolved by orbital studies is why elliptical galaxies are so nearly ellipsoidal in shape. The class of integrable potentials includes mass models with isodensity contours that are strongly nonellipsoidal, both "disky" and "boxy" (20). Triaxial models formed during the N-body evolution of nonequilibrium or unstable initial conditions are often found to be extremely boxy, even peanut-shaped (21, 22). The ubiquity of boxy models is perhaps not surprising in consideration of the fact that most orbits in triaxial potentials have strongly flaring shapes, as shown in Fig. 2 (23). In contrast, however, real elliptical galaxies have isophotal shapes that are almost precisely elliptical: the deviations are typically less than 1% and are as often in the direction of diskiness as of boxiness (24). It is unclear what, if anything, this fact might be telling us about the formation history of elliptical galaxies (25).

Stability

Any proposed stellar-dynamical configuration should be tested for stability before being accepted as a model of a real galaxy. The importance of instabilities in rapidly rotating stellar systems, such as spiral galaxies, has been recognized for at least three decades; this work has antecedents in the classic papers of Jacobi, Dirichlet, and others on the equilibrium and stability of rotating fluid bodies. In contrast, the importance of instabilities in galaxy models with little or no rotation was appreciated only quite recently. In part, the reason was that the stellar velocity vectors in a hot stellar system are pointed in all directions, so a perturbation in the stellar density might be expected to rapidly attenuate as the stars move along their respective orbits. However, it turns out that the stellar motions in a variety of interesting hot models are sufficiently correlated to induce instability, and these instabilities often grow on a time scale that is short compared to the age of the universe.

The first such instability to be discovered, and probably still the most important, is the so-called "radial-orbit instability" (26). A spherical model in which the orbits are strongly eccentric, or radial, evolves quickly into a triaxial bar. The instability arises because highly eccentric orbits are nearly closed, and nonradial forces can realign them around a barlike perturbation. The radial-orbit instability implies an upper limit to the degree of velocity anisotropy in spherical or spheroidal models and can sometimes be used to constrain the dynamical state of observed galaxies (27). This instability may also play a role in galaxy formation because collapse tends to result in eccentric orbits. The N-body simulations of such systems verify that collapse from a wide range of initial conditions leads to final states whose axis ratios are determined by a process similar to the radial-orbit instability (28).

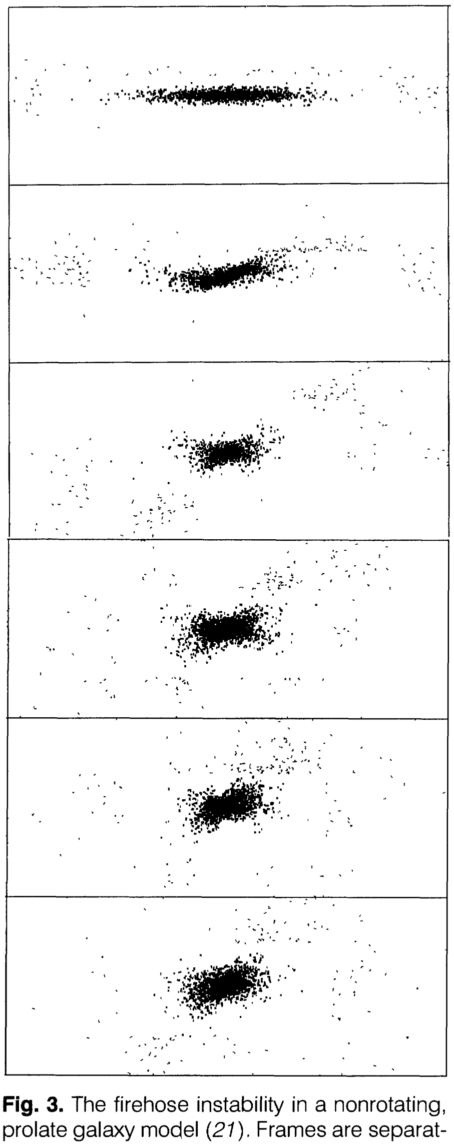

Fig. 3. The firehose instability in a nonrotating, prolate galaxy model (21). Frames are separated by about two crossing times. The final model is substantially more "boxy" than real elliptical galaxies; the simulation was not carried far enough to determine whether this boxiness would persist for a time corresponding to the age of the universe. This N-body simulation contained 10^4 particles and was executed on the Cray Y-MP at the University of Pittsburgh.

Another important instability is illustrated in Fig. 3. Models that are sufficiently elongated are unstable to global bending modes, which arise when the centrifugal force experienced by a star that moves along the bend overwhelms the gravitational restoring force (29). This instability is similar to the "firehose" instability of plasma physics (30) and is usually called by the same name. The N-body integrations of a family of prolate models suggest that galaxies with axis ratios more extreme than about 3:1 are unstable to these bending modes, which quickly act to "puff up" the short axis and reduce the flattening (21). This instability is a plausible explanation for why elliptical galaxies never have Hubble types more extreme than E7 (31). The firehose instability may also play a crucial role in the formation of the central bulges of spiral galaxies: N-body integrations of rapidly rotating disks show that the central regions, after forming a rotating bar through an instability in the disk plane, immediately bend out of the plane, producing a hot, spheroidal system with properties similar to those of galactic bulges (32).

Little is known yet about the stability of triaxial models.

Dark Matter and Central Black Holes

The model building described above is based on the premise that the gravitational potential in which the stars move is generated by the stars themselves. There is good reason to suspect, however, that much of the matter in the universe is dark (33). The most secure evidence for dark matter comes from studies of the rotation curves of spiral galaxies. The stars in these galaxies lie mainly in a thin disk and travel on nearly circular orbits around the galactic center. By comparing the observed centripetal acceleration, v_c^2/r, with the calculated gravitational acceleration attributable to the luminous mass, one infers that stars make up less than 10% of the total mass in spiral galaxies and that the dark matter is more extended spatially than the stars, in what is commonly called a dark halo. The shapes of these halos are not well constrained observationally but are thought to be spherical or ellipsoidal (34). The composition of the dark matter is a completely open question.

It is reasonable to expect that elliptical galaxies are also surrounded by extended dark halos. Their presence is more difficult to infer than in the case of spiral galaxies, however, because elliptical galaxies are not spherical and the orbits in elliptical galaxies are more complex than those in spiral galaxies. The basic principle behind the estimation of the gravitational potential in an elliptical galaxy can be illustrated by the case of a spherical and nonrotating galaxy in which the stellar velocities are everywhere isotropic. The first velocity moment of the collisionless Boltzmann equation yields an equation similar to the equation of hydrostatic equilibrium

$$\frac{d(\nu\sigma^2)}{dr} = -\nu\frac{d\Phi}{dr} \qquad (2)$$

Here $\nu(r)$ is the number density of stars at radius r, $\sigma^2(r)$ is their mean square velocity in one dimension, and $\Phi(r)$ is the gravitational potential, which may result from both stars and dark matter. Observations yield the projected number density and velocity-dispersion profiles from which $\nu(r)$ and $\sigma(r)$ can be inferred. Equation 2 then gives the potential $\Phi(r)$ and, hence, the mass $M(r) = G^{-1}r^2d\Phi/dr$, where G is the gravitational constant. This sort of analysis, applied to stars and other kinematical samples, suggests that dark matter may be present around at least some elliptical galaxies (*35*). But even in a spherical galaxy, the distribution of stellar velocities need not be isotropic, which leads to large formal uncertainties in the estimated $\Phi(r)$ (*36, 37*). If we permit the postulated dark-matter halos to be nonspherical, the uncertainties increase even more.

Recent work suggests that it is possible to map dark matter in hot stellar systems with the use of the full distribution of line-of-sight velocities of some test sample at a number of different positions, rather than just the velocity dispersion that appears in Eq. 2 (*38*). One way to obtain this additional information is to measure the Doppler-shifted velocities of a very large sample of discrete objects (stars, planetary nebulae, globular clusters, and so on) that surround the galaxy and construct a velocity histogram at a variety of radii. Such observations are greatly facilitated by a new generation of detectors that use optical fibers to observe hundreds of objects at once (*39*). Another route is to measure the integrated spectrum of the stars at a given point in a galaxy and ask what velocity distribution function would yield the observed line shapes after convolution with the spectrum of a single star (*40*). Such studies are only beginning but have the potential to place much stronger constraints on the dynamical state of elliptical galaxies than was possible in the past.

Another piece of evidence for dark matter in elliptical galaxies comes from kinematical studies of galactic nuclei. In 1978, a group demonstrated the possible existence of a black hole or other compact object with a mass of about 5×10^9 solar masses at the center of the giant elliptical galaxy M87 (*41*). The evidence was a moderately steep rise in the stellar velocity dispersion in the inner kiloparsec of the galaxy, which implied, with the use of Eq. 2, a large central mass. Very soon after, a number of workers stressed the nonuniqueness of this model because of uncertainties in the orbital distribution, and the most recent work suggests that no strong case can be made for a significant nonluminous component in the nucleus of this galaxy (*42*). However, it has long been clear that dead quasars should be roughly as common as giant galaxies, and it is natural to expect that the massive black holes that are thought to power the quasars should reside in galactic nuclei. This argument has prompted a search for dynamical signatures of massive black holes in other galaxies. To date, reasonably strong kinematic evidence for central mass concentrations has been obtained for a handful of nearby galaxies (*43*). The most convincing cases are the central bulges of spiral galaxies, which often exhibit a rapid rise in the velocity of stellar streaming very near the center. These nuclei are probably highly flattened and may even consist of disks of stars that formed from gas that fell in from much larger radii. However, the resolution limits imposed by the Earth's atmosphere have so far precluded the unambiguous determination of the shapes of these nuclei, and the resulting uncertainties in the inferred black hole masses are generally comparable to the masses themselves (*44*). It is not clear whether improvement of this situation is possible with ground-based instruments (*45*).

Formation of Elliptical Galaxies

One remarkable regularity in the morphology of elliptical galaxies is the form of their surface brightness profiles, that is, the dependence of the stellar number density on radius. Aside from scaling and shape factors, this dependence is very nearly the same for all elliptical galaxies. One commonly used parametrization is the so-called "$r^{1/4}$ law"

$$I(R) \propto \exp\left[-7.67\,(R/R_e)^{0.25}\right] \qquad (3)$$

first proposed by de Vaucouleurs in 1948 (*46*); here $I(R)$ is the luminosity density on the plane of the sky at apparent radius R, and R_e is the projected radius containing one-half of the total light. Equation 3 is accurate over at least three decades in radius for many bright elliptical galaxies (*47*). Although the detailed form of Eq. 3 is almost certainly without physical significance, the existence of a characteristic density law cries out for an explanation. Attempts to derive the $r^{1/4}$ law from a maximum-entropy principle have been largely unsuccessful (*48*). Nevertheless, computer simulations of galaxy formation demonstrate that a density profile similar to Eq. 3 is a very common outcome, as long as the evolution satisfies two general conditions: (i) there must be strong exchange of energy between the stars and the overall gravitational field, so that the initial conditions are forgotten, and (ii) the initial state must contain an appreciable fraction of matter at high phase-space densities, to allow the formation of a high-density core (*49*). One simple formation scenario that has received much attention is collapse from a cold, that is, low velocity dispersion, initial state. Computer simulations (*28*), as well as simple analytical arguments (*50*), confirm that such collapses produce galaxies with density profiles very similar to the $r^{1/4}$ law. However, it is clear from the numerical simulations that many other models of galaxy formation can produce good $r^{1/4}$ laws (*51*), and it does not seem especially likely that real galaxies formed through cold collapse.

One model for the formation of elliptical galaxies that has gained many adherents in recent years is the merger hypothesis proposed by Toomre (*52*). According to this hypothesis, most or all elliptical galaxies formed from the coalescence of spiral galaxies, either in the distant past or more recently. The initial motivation for the merger hypothesis was the observation that the rate of apparent merger events in the current universe is large enough to have formed all the observed elliptical galaxies over the course of 10^{10} years (*52*). This argument was strengthened by the discovery of dark-matter halos around spiral galaxies, which imply a much higher probability for a merger than if galaxies were no bigger than their luminous parts (*53*). More recent evidence in favor of the merger hypothesis includes the discovery of a number of fine-structure features, including ripples, twists, imbedded disks, and tidal tails, that may be indicative of past mergers (*54*). A large fraction of the elliptical galaxies that are examined closely seem to have such fine structure (*55*). Although the interpretation of this fine structure is still a matter of debate, it seems clear at least that elliptical galaxies are far less uniform in their structure and formation histories than was previously thought.

REFERENCES AND NOTES

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2. Many galaxies classified as E6 or E7 are probably disk galaxies.
3. With the assumption of random orientations, the distribution of intrinsic elongations can be estimated from a sufficiently large sample of galaxies [E. Hubble, *Astrophys. J.* **64**, 321 (1926)] and is only weakly dependent on the assumed geometry. One finds that most elliptical galaxies have axis ratios of about 3:5 and that spherical galaxies are rare [for example, D. Merritt, in *Morphological and Physical Classification of Galaxies*, G. Longo, M. Capaccioli, G. Busarello, Eds. (Kluwer Academic, Norwell, MA, 1992), p. 309].

4. Such a view of elliptical galaxies is implicit in some texts, for instance, K. F. Ogorodnikov, *Dynamics of Stellar Systems* (Pergamon, Oxford, 1965), among others.
5. F. Bertola and M. Capaccioli, *Astrophys. J.* **200**, 439 (1975); J. J. Binney, *Mon. Not. R. Astron. Soc.* **183**, 501 (1978).
6. It is difficult to understand in retrospect why the observation of low rotational velocities was very surprising because it had been known for many years that nonrotating oblate models could be constructed with any degree of flattening with the use of the two classical integrals in axisymmetric potentials [see, for example, G. G. Kuzmin and S. A. Kutuzov, *Bull. Abastumani Astrophys. Obs.* **27**, 82 (1962); D. Lynden-Bell, *Mon. Not. R. Astron. Soc.* **123**, 447 (1962)].
7. The equations that describe the evolution of a nonneutral plasma free of magnetic fields are more similar to those of self-gravitating systems; see R. C. Davidson, *Physics of Nonneutral Plasmas* (Addison-Wesley, Reading, MA, 1990).
8. Most treatments of plasma phenomena are only valid on length scales that are long compared to the Debye length, scales at which it is common to assume that the plasma is homogeneous. The gravitational analog to the plasma Debye length is the Jeans length, which is roughly equal to the size of a galaxy. This is one reason why the results of plasma physics cannot usually be carried over into galactic dynamics.
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14. T. S. Statler, *ibid.* **321**, 113 (1987).
15. S. Aarseth and J. J. Binney, *Mon. Not. R. Astron. Soc.* **185**, 227 (1978); R. H. Miller, *Astrophys. J.* **223**, 122 (1978); A. Wilkinson and R. A. James, *Mon. Not. R. Astron. Soc.* **199**, 171 (1982); J. Barnes, *Astrophys. J.* **393**, 484 (1992).
16. Triaxiality has certain observable consequences, such as an apparent twisting of the isophotes in galaxies for which the axis ratios vary with radius. The observational evidence for triaxiality is only moderately convincing; see the reviews by F. Bertola [in *Morphological and Physical Classification of Galaxies*, G. Longo, M. Capaccioli, G. Busarello, Eds. (Kluwer Academic, Norwell, MA, 1992), pp. 115–125], M. Franx [in *ibid.*, pp. 23–38], and D. Merritt [in *ibid.*, pp. 309–320].
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34. One way to constrain the shapes of dark halos is through studies of the so-called polar-ring galaxies, disk galaxies that have a ring of gas orbiting in a plane perpendicular to that of the stellar disk. By comparing the orbital velocities of the stars and gas in the two disks, one can place limits on the flattening of the gravitational potential that is produced by the dark halo. The most careful study to date of a polar-ring galaxy [P. Sackett and L. Sparke, *Astrophys. J.* **361**, 408 (1990)] concludes that only weak constraints can be placed on the elongation of its halo. Simulations of the formation of structure in the universe, based on the cold, dark-matter hypothesis, suggest that most dark halos should be nearly prolate and very elongated [J. Dubinski and R. Carlberg, *Astrophys. J.* **378**, 496 (1991)].
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37. A way to avoid the problem of velocity anisotropy is for one to use hot gas to gauge the gravitational potential, by measuring the dependence of gas pressure on radius and applying the equation of hydrostatic equilibrium. Instrumental shortcomings have so far strongly limited the efficacy of this technique [G. Fabbiano, *Annu. Rev. Astron. Astrophys.* **27**, 87 (1989)].
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43. ———, *ibid.* **324**, 701 (1988); J. Kormendy, *ibid.* **335**, 40 (1988); ——— and D. O. Richstone, *ibid.* **393**, 559 (1992).
44. O. E. Gerhard, in *Physics of Active Galactic Nuclei*, W. J. Duschl and S. J. Wagner, Eds. (Springer-Verlag, Berlin, 1992), p. 273.
45. The Hubble Space Telescope was supposed to have settled this question by providing kinematical information, at very high spatial resolution, about the nuclei of nearby galaxies. The spherical aberration in the primary mirror has so far precluded such observations. Aberration-corrected images from the Hubble Space Telescope do provide somewhat higher resolution than ground-based telescopes, and, with the combination of stellar luminosity profiles from the Hubble Space Telescope and ground-based kinematical data, the case for massive nuclear black holes in two galaxies can be strengthened somewhat [T. Lauer *et al.*, *Astron. J.* **103**, 703 (1992); T. Lauer *et al.*, *ibid.* **104**, 552 (1992)].
46. G. de Vaucouleurs, *Ann. Astrophys.* **11**, 247 (1948).
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48. The Boltzmann entropy, $S = -\int f \ln f d^3x d^3v$, is precisely conserved in a collisionless stellar system. It is nevertheless common to assume that something like the entropy is maximized during the chaotic period of galaxy formation. However, the maximization of S subject to fixed values of the total mass and energy leads to a stellar distribution function that is incompatible with finite mass and energy [(*4*), p. 302]; thus, stellar systems admit no configuration of stationary entropy.
49. T. S. van Albada, *Mon. Not. R. Astron. Soc.* **201**, 939 (1982); A. May and T. S. van Albada, *ibid.* **209**, 15 (1984). The "charming" oxymoron violent relaxation was coined by D. Lynden-Bell [*ibid.* **136**, 101 (1967)] to describe the formation of a galaxy in a rapidly varying gravitational field.
50. D. Merritt, S. D. Tremaine, D. Johnstone, *ibid.* **236**, 829 (1989); D. Spergel and L. Hernquist, *Astrophys. J.* **397**, L75 (1992).
51. Examples are (*21*); (*49*); R. B. Larson, *Mon. Not. R. Astron. Soc.* **166**, 585 (1974); and D. Merritt and M. Stiavelli, *Astrophys. J.* **358**, 399 (1990).
52. A. Toomre, in *The Evolution of Galaxies and Stellar Populations*, B. M. Tinsley and R. B. Larson, Eds. (Yale University Observatory, New Haven, CT, 1977), p. 401. Perhaps the first clear statement of the merger hypothesis was in A. Toomre and J. Toomre, *Astrophys. J.* **178**, 623 (1972).
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54. F. Schweizer, *Astrophys. J.* **252**, 455 (1982); D. F. Malin and D. Carter, *ibid.* **272**, 534 (1983); R. Bender, *Astron. Astrophys.* **202**, L5 (1988).
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