The Fractional Quantum Hall Effect

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Recent research has uncovered a fascinating quantum liquid made up solely of electrons confined to a plane surface. Found only at temperatures near absolute zero and in extremely strong magnetic fields, this liquid can flow without friction. The excited states of this liquid consist of peculiar particle-like objects that carry an exact fraction of an electron charge. Called quasiparticles, these excitations can themselves condense into new liquid states. Each such liquid is characterized by a fractional quantum number that is directly observable in a simple electrical measurement. This article attempts to convey the qualitative essence of this still unfolding phenomenon, known as the fractional quantum Hall effect.

THE COLLECTIVE BEHAVIOR OF THE MANY ATOMS OR MOLEcules in a macroscopic system is a fundamental issue in modern physics. The periodic solid and the shapeless liquid are condensed forms of matter, distinguished from the gaseous state by the strong collective interactions of their constituent atoms. Of particular interest are those condensed systems whose macroscopic behavior is dominated by the laws of quantum mechanics. Such systems, in which the quantum uncertainty in the positions of the constituent particles exceeds their separation, often exhibit bizarre properties. Superconductors are notable examples; these materials can carry electrical current without any dissipation of energy. Less well known are the superfluids, which exhibit frictionless flow and other peculiar properties like quantum whirlpools. These unusual effects are examples of macroscopic quantum phenomena, belying the notion that quantum mechanics concerns only the atomic world.

Physical systems of reduced dimensionality, in which the particles are confined to a plane or line rather than occupying three-dimensional (3-D) space, have recently become subjects of intense scrutiny. Most often these systems are artificially fabricated from semiconductor crystals. While their great interest lies partly in real and potential electronics applications, they are equally fascinating from the pure physics point of view. Beyond providing an ideal testing ground for modern theories of condensed systems, these man-made structures have revealed totally new physical phenomena. Preeminent among these is the fractional quantum Hall effect; a macroscopic quantum phenomenon that is the result of the condensation of a collection of electrons into a bizarre fluid state.

A two-dimensional (2-D) system of electrons is surely one of the simplest many-particle systems imaginable. Add a magnetic field to it and a fascinating microcosm unfolds. An electron quantum liquid, unlike any other existing liquid, is created. Near absolute zero this liquid flows without dissipation, circumventing obstacles in the plane. Simple electrical measurements reveal the so-called Hall resistance to be quantized to exact rational multiples of a universal constant. A slight increase in temperature creates peculiar particle-like objects in the liquid that carry a precise fraction of the charge of an electron. These quasiparticles themselves can condense into liquid states, leading to a hierarchy of parent and daughter fluids. Very recently the spin of the electron has been found to further enrich the spectrum of phenomena. On the horizon lies the possibility that these strange liquids can somehow freeze into electron solids with new properties as yet unseen.

The first glimpse of this intriguing microscopic world was provided by the discovery (*1*) of the fractional quantum Hall effect in 1982. Since then much progress has been made toward a theoretical understanding of the underlying physics and many new experimental observations have been made (*2*). Our article attempts to convey the qualitative essence of this new many-particle phenomenon and to highlight those aspects that remain enigmatic.

Preliminaries

There are no truly 2-D systems in nature. Like a game of billiards, however, some are well approximated by a 2-D model. In the same way we can construct only approximately 2-D systems of electrons. Nowadays the best such construction confines a pool of electrons to the interface between two ideally matched semiconductor crystals. It is a fascinating reality, due to quantum mechanics, that if a perfect crystal could be grown, without impurities or defects, an electron could move through it without resistance at zero temperature. Its wave-like nature allows it to flow through the crystal lattice of atoms without collisions. The same is true at the interface between two crystals, provided they are perfectly matched. The best such systems are fabricated from the semiconductors gallium arsenide (GaAs) and gallium aluminum arsenide (GaAlAs) which are grown in thin layers atop a suitable substrate. Excess electrons, donated by remote impurities, find their way to the interface and are bound there by the different chemical nature of the two semiconductors. After donating their electrons, the impurity atoms are left positively charged, the net charge of the sample thus being zero. Typical samples contain some 10^{11} electrons per square centimeter, corresponding to a mean spacing of a few hundred angstroms. The interfacial binding does not restrict the electrons from moving in the plane. In fact, at present the best such samples (*3*) allow electrons to move the huge distance of about 0.1 mm in the 2-D plane, passing some 400,000 atoms without suffering a severe collision. Such freedom is only obtained at temperatures near absolute zero where the crystalline vibrations—really a type of "imperfection"—are minimized.

The simplest way to probe the properties of any system of freely moving electrons is to measure their electrical properties. These so-called transport measurements have provided essentially all that we

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know about the fractional quantum Hall effect. To make such measurements a small "chip" of the layered semiconductor sample, typically a few millimeters on a side, is processed so that the region containing the 2-D electrons has a well-defined geometry. The frequently used "Hall bar" geometry is depicted in Fig. 1. A tiny electrical current is driven along the central section of the bar, while the various side arms serve as probes to measure the induced voltages. Two probe configurations are important: the longitudinal voltage difference V between two probes on the same side of the central bar, and the so-called Hall voltage V_H between probes situated opposite one another across the bar. We usually convert these voltages into resistances by dividing by the current I running down the bar. The longitudinal resistance R has the same significance as one's conventional notion of the electrical resistance of an ordinary material. Its magnitude is a measure of the frequency of collisions suffered by the electrons. The Hall resistance R_H, however, is different.

Discovered 120 years ago, the Hall resistance is one of the most frequently measured quantities in solid-state physics. R_H is zero in the absence of a magnetic field. When a field is applied perpendicular to the 2-D plane the magnetic force causes the moving charges to accumulate at one side of the bar. This continues until the electric force that results from the charge separation exactly cancels the magnetic force. A classical analysis yields the simple result:

$$R_H = B/Ne \qquad (1)$$

where B is the magnetic field, N the number of charges per unit area in the plane, and e the charge of an electron. Thus, a Hall measurement establishes N, the carrier concentration. Only a decade ago this simple result was expected to remain valid in very high magnetic fields and at the lowest temperatures. Figure 2 shows both the Hall resistance R_H and the longitudinal resistance R in a 2-D electron sample as functions of magnetic field. Obtaining such data

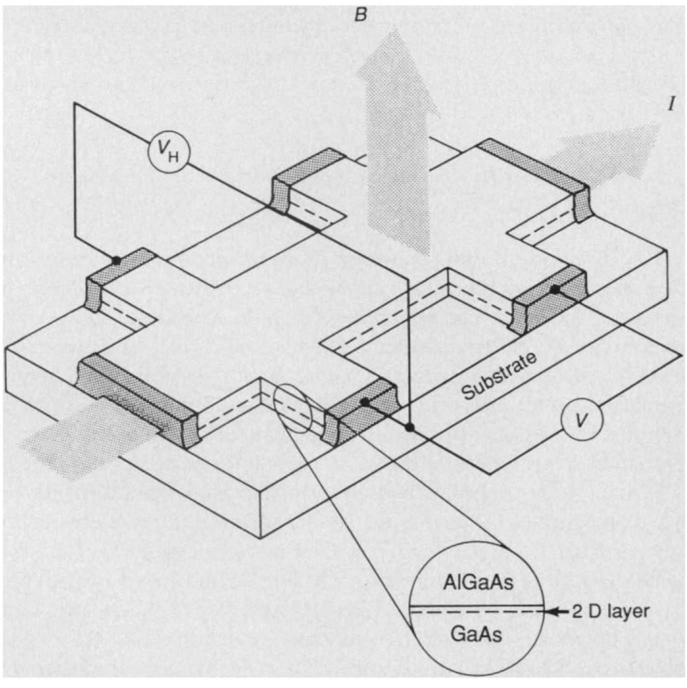

Fig. 1. A typical Hall bar sample. The structure is formed by chemically etching away unwanted material. The dotted line indicates the 2-D electron gas at the interface between gallium arsenide (GaAs) and aluminum gallium arsenide (AlGaAs). The magnetic field B and electrical current I are shown, as are the longitudinal and Hall voltages, V and V_H, respectively. The shaded regions at the ends of each arm of the bar are where electrical contact is made to the 2-D electron gas.

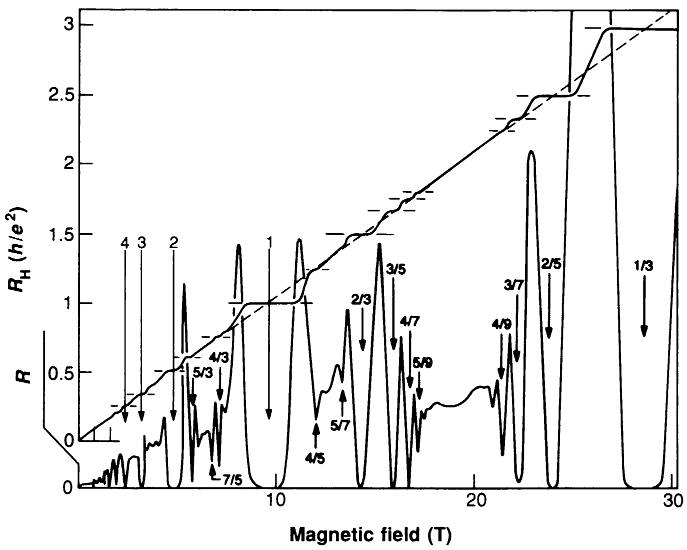

Fig. 2. Composite view showing the Hall resistance R_H and longitudinal resistance R of a 2-D electron gas versus magnetic field. The diagonal dashed line passing through the R_H trace represents the classically expected Hall resistance for this sample. For each of the plateaus in R_H there is an associated minimum in R. The numbers give the value of p/q determined from the value of R_H on the plateaus. While some of the p/q values are integers, the great majority are fractions. Note in particular the "1/3 state" at the far right. This most prominent example of the fractional quantum Hall effect exhibits a Hall plateau at $R_H = (h/e^2)/(1/3) = 3h/e^2$.

is difficult; not only does the sample represent state-of-the-art crystal growth, but the magnetic fields (up to 30 T) and temperatures (often as low as 0.02 K) are extreme. The diagonal dotted line represents the simple result expected from Eq. 1. Obviously, 2-D electrons in high magnetic fields were not at all understood 10 years ago.

There are two astonishing aspects to Fig. 2. While oscillations in the longitudinal resistance R were anticipated, that it would fall essentially to zero over wide ranges of magnetic field was totally unexpected. The second aspect, perhaps even more amazing than the first, are the plateaus in the Hall resistance R_H. Close examination of the values of R_H at these plateaus reveals that all can be described by a universal formula:

$$R_H = (h/e^2)/(p/q) \qquad (2)$$

This expression depends only on the ratio of fundamental constants; the Planck constant h and the electronic charge e. The numbers p and q are simply integers. These quantized values are totally independent of the sample specifics. The plateaus in R_H and zeros in R, known collectively as the quantum Hall effect, are clear signatures of hitherto unappreciated aspects of 2-D electron systems.

The subset of plateaus for which the ratio $p/q = 1,2,3$ . . . is an integer was discovered (*4*) before the first fractional value $p/q = 1/3$ was found. We now know that the two cases reflect very different physics. The integer case can be understood solely in terms of individual electrons in a magnetic field. The fractional p/q values are far more subtle, reflecting entirely new physics arising from the collective behavior of all the electrons.

The essential ingredient for understanding the integer quantum Hall effect (IQHE) has been known for more than 50 years. It is the quantization of the circular motion of a charged particle in the presence of a magnetic field. Classically, an electron moves in a circular orbit perpendicular to the magnetic field. Any radius is allowed, only the period of revolution is fixed by the magnetic field strength, B. Quantum mechanics, however, demands discrete values of the radius, in the same way as it enforces discrete Bohr orbits on an atom. Like the Bohr atom, these discrete orbits correspond to discrete energy levels, called Landau levels. The Landau levels are spaced equally by an amount called the cyclotron energy which is proportional to the magnetic field B. Thus, for a system of electrons confined to a 2-D plane, the entire energy spectrum consists of a ladder of discrete Landau levels with wide energy gaps in between. For neither 3-D nor 1-D systems do similar gaps exist. These gaps are at the heart of the integer quantum Hall effect.

Each Landau level can accommodate a large number of electrons, all at the same energy, because it is possible to place the center of each orbit at many equivalent places in the 2-D plane. Since the size of the orbits decrease with increasing magnetic field (radius about 80 Å at 10 T), this so-called "degeneracy" of the Landau levels increases with field. In fact, every Landau level can accommodate $D = eB/h$ electrons per unit sample area; about 2.4×10^{11} per square centimeter at 10 T. This is already a remarkable result since it is independent of all sample parameters.

Figure 3 illustrates these concepts. Dividing the number of electrons per unit sample area, N, by the degeneracy D of the Landau levels, defines the "filling factor" $\nu = N/D$. This quantity tells how many Landau levels are occupied. At very high magnetic field the degeneracy D exceeds N, all electrons lie in the lowest Landau level and $\nu < 1$. On reducing the field two things happen: The spacing in energy between Landau levels, as well as their degeneracy D, decreases. A magnetic field B_1 is reached for which the lowest level is exactly filled, that is $D = eB_1/h = N$ and the filling factor is $\nu = 1$. Further reduction of the field forces some electrons up into the second Landau level. Eventually a field B_2 is reached where the two lowest levels are exactly filled, $D = eB_2/h = N/2$ and the filling factor is $\nu = 2$. For any integer j there is a field $B_j = Nh/je$ at which the j lowest Landau levels are exactly filled and all higher levels are empty. Let us evaluate the Hall resistance R_H at these special magnetic fields. First, using Eq. 1 and the definition of D, we can express R_H in terms of the filling factor:

$$R_H = B/Ne = (hD/e)/Ne = (h/e^2)/\nu \quad (3)$$

At the special fields B_j the filling factor ν equals the integer j giving:

$$R_H = h/je^2 \quad (4)$$

These are exactly the values of the integer quantum Hall plateaus! We can even understand the vanishing of the longitudinal resistance R at these fields. Zero resistance implies no energy dissipation. Dissipation only occurs if electrons can easily scatter into empty energy levels. At the special fields B_j the nearest empty states are at much higher energy across the Landau gap. At low temperatures these states cannot be reached and thus dissipation cannot occur.

Is this all there is to the IQHE? A moment's thought reveals a serious problem with this simple picture. Our solution only works for the precise field values B_j. How can R_H remain flat over wide stretches of magnetic field? This is a formidable question and its solution (*5*) represents the second fundamental ingredient of our understanding the IQHE. The missing element is the residual imperfection inherent in any real sample. There are always some impurities or defects remaining in the sample despite one's best efforts. These imperfections can trap some of the 2-D electrons and prevent them from participating in the current flow. Slight departures of the magnetic field from the special values B_j merely changes the number of these trapped electrons but not the number of occupied Landau levels. This causes no change in the resistances R and R_H which reflect only the nontrapped, current-carrying electrons. Larger magnetic field shifts overwhelm the capacity of the traps and thereby change the number of occupied Landau levels and thus the resistances. One is led to a paradoxical truth: the existence

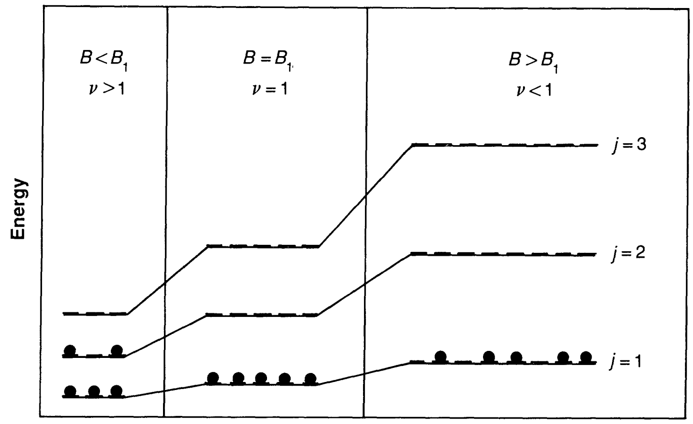

Fig. 3. Three lowest Landau levels, $j = 1,2,3$, in a five-electron system. Each panel corresponds to a specific magnetic field, B. The number of available states within each level is indicated. In the right-hand panel the magnetic field is high enough so that all five electrons may reside in the lowest level. In the middle panel the field has been reduced to the value B_1 where the lowest level is completely occupied and all higher levels are empty. This corresponds to the filling fraction $\nu = 1$. In the left panel the field has been further reduced, forcing some electrons into the $j = 2$ Landau level.

of the plateaus requires imperfections in the sample while the value of R_H on the plateau is a universal constant. Were the sample truly perfect the plateaus would be absent and R_H would return to the straight classical line!

What about the fractional plateaus, which actually dominate Fig. 2? Are they explained by some simple extension of the above argument? The answer is an unequivocal "no." We have argued that the integer plateaus are the result of gaps in the energy spectrum. Since the phenomena of the fractional effect appear the same as the integer case, we are led to search for additional energy gaps. Considering each electron individually leads only to the Landau gaps associated with integer values of the filling factor ν; there are no gaps at fractional values of ν. The fractional quantum Hall effect (FQHE) must result from some new collective state in which all electrons participate.

The Standard Model

Any description of the collective motion of many particles has to take into account the forces acting between them. In the case of electrons, this is the familiar coulomb repulsion of like charges. The motion of each electron depends, through this force, on the motion of all other electrons, especially those nearby. Furthermore, as we are dealing with electrons interacting on a microscopic scale, the notions of classical physics are inadequate and the inherently probabilistic principles of quantum mechanics must be considered. The final result of applying these principles is a wave function Ψ whose magnitude gives the probability for finding the electrons in any particular configuration. For the most prominent FQHE state, at filling factor $\nu = 1/3$, a remarkably simple, and nearly exact wave function has been obtained. This ingenious result, due to Laughlin (*6*), provides the basis for the standard model of the FQHE. Our objective is to qualitatively illustrate this wave function and thereby the electronic configuration underlying the FQHE.

Even with Laughlin's wave function in hand, we are confronted with the difficulty of illustrating a function that depends on the positions of many particles with only a single picture. To accomplish this, we make the great simplification of imagining a "snapshot" in which all of the electrons in the sample, save one, are held in fixed positions. The remaining electron, which we henceforth call the "representative," will be described by a smooth landscape whose elevation denotes the probability for finding this electron at a given location. This picture is thus a mixture of classical and quantum concepts and is not strictly correct. In reality, all electrons should be treated equally. This means any electron could be chosen as the representative. It also must be kept in mind that the companion electrons are not fixed in position and our "snapshot" is merely one frame of a larger film. On average the total electron distribution is really completely uniform.

We begin our illustration by stepping back to the simplest situation, for which the system has only one electron. In this case our earlier description of quantized circular orbits should be valid, and we can assume the single electron lies in the lowest Landau energy level. Since we do not know where the electron is in the 2-D plane, we cannot locate the center of its cyclotron orbit. The probability of finding the electron is then completely uniform across the 2-D plane, just as it would be if there were no magnetic field at all. How then does the magnetic field influence the probability distribution?

Associated with the magnetic field are so-called "flux quanta." In some sense these are the quantum counterparts of the classical notion of magnetic flux lines. While classical physics insists these lines themselves have no reality, in the quantum world they are more tangible. In fact, the regular array of flux lines trapped in a superconductor has been observed by various techniques. As with electrons, quantum mechanics requires uncertainty in the position of the flux quanta. Thus, just as a uniform charge density can result from a collection of discrete electrons, so a uniform magnetic field derives from a collection of discrete flux quanta. The magnitude of the flux quantum $\Phi_0 = h/e = 4.1 \times 10^{-7}$ G cm^2 is tiny by ordinary standards. The earth's small magnetic field of 0.3 G corresponds to almost 10^6 flux quanta per square centimeter. Far higher flux densities than this are required for observation of the FQHE.

These flux quanta associated with the magnetic field create tiny vortex-like dimples in the probability distribution of our representative electron. As depicted in Fig. 4A, at the center of these vortices the probability of finding the electron is zero. How can this distribution be regarded as uniform? The answer to this lies in the huge degeneracy of the Landau level that we have already encountered. There are many equivalent ways to distribute the vortices around in the 2-D plane, Fig. 4A represents just a specific choice. On the average, the probability for finding the electron is again completely uniform. Only when additional electrons are added to the system is this indeterminacy in the vortex positions tempered. As we will see, the FQHE arises from an unusually strong correlation between the positions of the electrons and the vortices.

The ground state at $\nu = 1/3$. We now would like to add electrons to our system. These additional electrons (red spheres in Fig. 4) will be placed in fixed positions and our probability distribution (green) will be that of the original "representative" electron. Again, any electron could be chosen as the representative and our illustration can only be thought of as a snapshot which belies the continual state of motion of all the electrons. On adding the first of the "companion" electrons we immediately confront a basic tenet of quantum mechanics: the Pauli exclusion principle. This requires that no two electrons may reside at the same position. Thus, we must put this second electron in a position avoided by the representative. From Fig. 4A, we see the obvious place is directly on one of the vortices associated with a flux quantum. All subsequent companion electrons must be placed onto unoccupied vortices. We can keep adding electrons until all available vortices are occupied. This situation, shown in Fig. 4B is clearly special; it corresponds to complete filling, $\nu = 1$, of the Landau level. Attempts to add more electrons requires placing them in higher Landau levels, at enormous energetic cost. We now have a special case, the fully filled Landau level, in which every electron has a single vortex attached to it. This association is entirely the result of the Pauli exclusion principle. For the FQHE the Landau level is only partially filled and there are more vortices than electrons. The Pauli principle does not require any specific distribution of the "extra" vortices. It is the repulsive interactions between the electrons, the heart of the FQHE, that creates a new, correlated, arrangement between all the vortices and the electrons.

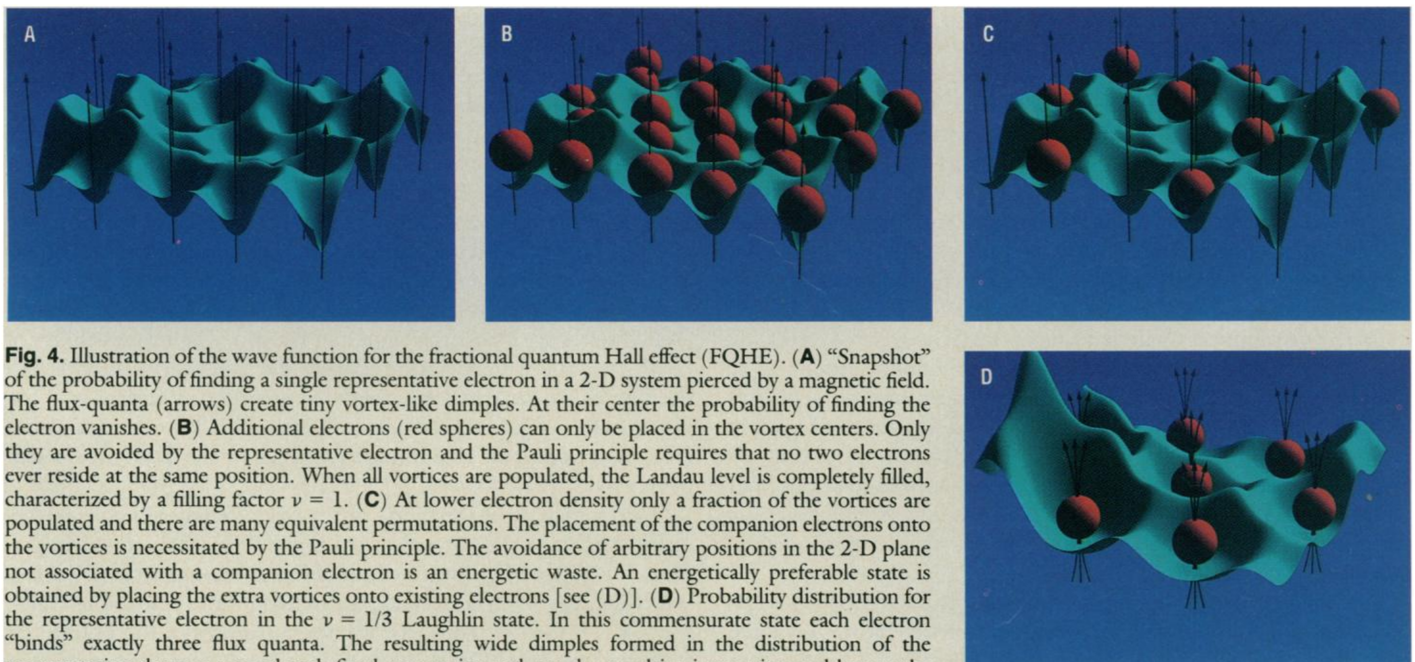

Fig. 4. Illustration of the wave function for the fractional quantum Hall effect (FQHE). (**A**) "Snapshot" of the probability of finding a single representative electron in a 2-D system pierced by a magnetic field. The flux-quanta (arrows) create tiny vortex-like dimples. At their center the probability of finding the electron vanishes. (**B**) Additional electrons (red spheres) can only be placed in the vortex centers. Only they are avoided by the representative electron and the Pauli principle requires that no two electrons ever reside at the same position. When all vortices are populated, the Landau level is completely filled, characterized by a filling factor $\nu = 1$. (**C**) At lower electron density only a fraction of the vortices are populated and there are many equivalent permutations. The placement of the companion electrons onto the vortices is necessitated by the Pauli principle. The avoidance of arbitrary positions in the 2-D plane not associated with a companion electron is an energetic waste. An energetically preferable state is obtained by placing the extra vortices onto existing electrons [see (D)]. (**D**) Probability distribution for the representative electron in the $\nu = 1/3$ Laughlin state. In this commensurate state each electron "binds" exactly three flux quanta. The resulting wide dimples formed in the distribution of the representative electron around each fixed companion reduces the repulsive interaction and lowers the total energy. In the real system in which all electrons are delocalized, each electron develops a threefold vortex around each of its companions.

To see these new correlations, we now decrease the number of electrons below the $\nu = 1$ condition. As depicted in Fig. 4C, there is now an excess of vortices over electrons. While the companion electrons must sit on vortices, owing to the Pauli principle, there are many equivalent distributions of the electrons among the vortices. The unoccupied vortices represent random positions that the representative electron avoids, to no energetic advantage. A far preferable arrangement is to place these empty vortices onto the existing electrons. Multiple vortices are larger than single ones and are therefore more strongly avoided by the representative. Since a companion electron sits at the center of each multiple vortex, the repulsive interactions with the representative are reduced, and along with it the total energy of the system.

A particularly favorable state is created when the number of flux quanta is a multiple of the number of electrons. Such a situation arises at filling factor $\nu = 1/3$ where there are three flux quanta for each electron. In this commensurate state, each electron sits in a large dimple and the total energy is significantly reduced. The situation is illustrated in Fig. 4D for the representative electron. Such a representation continues to hold true in the actual many-particle state in which all electrons create threefold vortices about all companion electrons. Similarly favorable situations should exist at filling factor $\nu = 1/5$, $1/7$, and so forth. As we will see, states in which an even number of quanta are bound to each electron are quantum mechanically not allowed.

All these $\nu = 1/m$ FQHE states have a beautifully simple mathematical representation first proposed by Laughlin (*6*). We denote the position (x_j, y_j) of each electron j in the 2-D plane by a complex number $z_j = (x_j - iy_j)$. Then, aside from an unimportant factor, the many-particle wave function for n electrons can be written as a simple product over all differences between particle positions $(z_j - z_k)$

$$\Psi(z_1, z_2, z_3, \ldots, z_n) = (z_1 - z_2)^m \times (z_1 - z_3)^m \times (z_2 - z_3)^m \times \ldots \times (z_j - z_k)^m \times \ldots (z_{n-1} - z_n)^m \qquad (5)$$

The square of the wave function $|\Psi|^2$ represents the probability of finding a configuration in which there is one electron at position z_1, another electron is at position z_2, a third electron at position z_3, and so forth.

This mathematical representation automatically obeys the Pauli principle. The probability of finding two electrons at the same site is zero since one of the factors on the right-hand side becomes zero. A more subtle property of Ψ is that if any two electrons are interchanged (such as $z_2 \leftrightarrow z_3$), Ψ will change its sign if m is an odd integer.

It will not change its sign if m is even. Quantum theory insists that if Ψ is to describe electrons, then it must change its sign under particle exchange. Thus Laughlin's wave function can hold only for filling factors $\nu = 1/m$ where m is odd. For the Laughlin ground states the distribution of electrons is optimally correlated, reducing the repulsive coulomb interaction to a minimum. Addition or subtraction of a single electron or flux quantum disturbs this inherent order at a considerable energetic cost. For this reason states at $\nu = 1/m$ are referred to as condensed many-particle ground states. Since the mutual electronic positions are not fixed as in a solid, but rather free like in a liquid, and since this freedom is of a quantum mechanical, rather than a classical nature, the term condensed quantum liquid applies.

Quasiparticles. The Laughlin ground state is an accurate description of the FQHE state only at absolute zero temperature and at the exact magnetic field for $\nu = 1/m$ filling. Departure from either condition results in the creation of defects, called quasiparticles, in the liquid state. Theory asserts that these defects carry fractional charge.

The charge $-e$ of an electron is the fundamental quantum of electric charge. No particle carrying a fraction of $-e$, has ever been directly observed. Even the famous quarks of high-energy physics, which are held to carry fractional charge, have not been found in isolation. The notion of quasiparticles charged to an exact rational fraction of e is, at first sight, a puzzling implication of the theory of the FQHE.

What are these quasiparticles? To be certain, our electrons do not dissociate into 3, 5, or 7 . . . identical pieces. Fractionally charged quasiparticles are a convenient theoretical concept. They describe the fact that this many-electron system is able to harbor defects that act as though they carry fractional charge. Removal and addition of charges to the total system, can only be performed in units of e. With the framework of illustrations developed in the last section, these quasiparticles can, in fact, be intuitively described.

Using again the $\nu = 1/3$ state as a concrete example, we recall that at exactly 1/3 filling all particles are condensed into a highly correlated many-particle ground state. This ground state is a uniformly charged 2-D electron liquid in which the negative charge of each electron exactly compensates for the charge depletion caused by the surrounding threefold vortex. A minute change in filling factor, slightly off $\nu = 1/3$, is not expected to destroy this condensed phase. The quantum fluid instead tries to remain condensed by creating a few defects in its fabric. To visualize these defects, imagine the removal of an electron from the 1/3 state in Fig. 4D. This leaves behind a threefold vortex effectively carrying a charge of $+e$. In the absence of the electron, the three surplus flux quanta are no longer bound together and, therefore, are able to drift apart, each one of them dragging with it a vortex in the representative's distribution. The charge deficiency in each vortex amounts then to exactly $+e/3$. These local depressions in the charge density are called quasi-holes. Similarly, one can imagine the absence of one flux quantum. This situation, while harder to visualize, corresponds to a negatively charged defect $(-e/3)$ called a quasi-electron. A number of recent experiments (*7*) have suggested that these fractionally charged quasiparticles may, in fact, be observable.

Existence of such quasiparticle defects in the parent quantum liquid disturbs the correlated motion of the condensed carriers. The introduction of each quasiparticle raises the energy of the system by a fixed amount. This finite energy threshold for the creation of quasiparticles represents the sought after gap in the energy spectrum of the quantum liquid.

The existence of mobile charged particles, a gap in the energy spectrum, and the presence of a small degree of imperfection, provides all the ingredients for the observation of a quantization in the Hall effect and vanishing longitudinal resistance, R. From the point of view of electrical transport, the condensed quantum liquid at exactly $\nu = 1/m$ filling, separated by a gap from its excited states, resembles the completely filled Landau level. There the nearest excited states are across the large Landau gap. The inaccessibility of these states at low temperature explains the vanishing resistance R in the IQHE. By exact analogy, we now expect R to vanish at $\nu = 1/m$ in the FQHE. The only difference is that much lower temperatures are required since the FQHE gaps are much smaller than the Landau gaps.

Slight variation of the filling factor from exactly $\nu = 1/m$ creates quasiparticles. Again in analogy to the IQHE, these initial excitations are trapped by imperfections. Hence, we expect R to remain zero and the Hall resistance to remain at its $\nu = 1/m$ value $R_H = h/\nu e^2 = mh/e^2$. Thus, the transport features of the FQHE are analogous to the IQHE. The fundamental new physics in the FQHE is the creation of a many-particle ground state separated

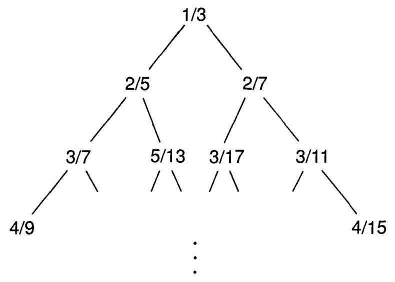

Fig. 5. The hierarchy of fractional states deriving from the primitive 1/3 state. The 2/5 and 2/7 states are the first daughters of the 1/3 state being formed from its quasi-electrons and quasi-holes respectively.

from its excitations by an energy gap. Without this no analogy could be made.

The magnitude of the energy gap is characteristic of each FQHE state. Apart from the condensation energy of the ground state, it is the single most important parameter of the quantum liquid and has been determined theoretically by a variety of different analytical and numerical schemes. This gap energy is also the quantity most accessible to experiment (*8*). Raising the temperature at exact fractional filling creates equal numbers of quasi-electrons and quasi-holes. These thermally created quasiparticles enhance the electrical conductivity of the system. The temperature dependence of the conductivity provides a measure of the energy gap. For the $\nu = 1/3$ liquid, the strongest and best understood of the FQHE states, the experimentally determined energy gap approaches the theoretically calculated value to within 20%. Considering the tremendous computational difficulties in deriving the theoretical gap value, this represents an astonishingly good agreement and a great success for the standard theoretical model of the FQHE.

The hierarchy. Laughlin's wave function, together with fractionally charged quasiparticles, provides an explanation for the FQHE at filling factor $\nu = 1/m$ with m an odd integer. A case can also be made for $\nu = (1 - 1/m) = 2/3, 4/5, 6/7, \ldots$ arguing that at such filling factors the Landau level is depleted by 1/3, 1/5, 1/7, . . . and condensed states develop among the holes in the distribution. However, many of the pronounced FQHE states, such as $\nu = 2/5$, 3/5, 3/7, 4/7, . . . are not included. The prevailing theoretical model regards these states as daughter states derived from the fluids at $1/m$. How does this come about? As the filling factor deviates considerably from exactly $1/m$, a large number of free quasiparticles are created in the quantum liquid. Being charged, these quasiparticles correlate their relative positions and try to stay optimally apart. At a critical density they themselves can condense into a correlated quantum liquid of quasiparticles. As an example, the FQHE at filling factor $\nu = 2/5$ is regarded as the many-particle daughter state condensed from $-e/3$-charged quasi-electrons of the $\nu = 1/3$ primitive state. The equivalent daughter state condensed from quasi-holes emerges at $\nu = 2/7$. Since daughter liquids develop quasiparticles of their own, the theoretical argument can be continued ad infinitum if not terminated by the formation of a yet unobserved quantum crystal.

Haldane showed how to arrange the resulting quantum-fluids into a hierarchy (*9*) of exclusively odd-denominator fractions that defines their line of descent. Figure 5 shows the first daughter states of the primitive $\nu = 1/3$ Laughlin liquid, several of which are visible in the experimental data of Fig. 2. The hierarchical scheme of daughter states provides a rationale for the existence of FQHE features at filling factor $\nu = p/q$ and orders the sequence of their appearance. However, compared to the Laughlin liquids at filling factor $\nu = 1/m$, very little is known about these higher order many-particle states. The theoretical calculations rapidly become intractable as one progresses down the hierarchy and experimental data on the energy gaps of daughter liquids have begun to emerge only recently.

The standard model seems to have established a satisfactory interpretation for the origin of the FQHE. Condensed quantum liquids at fractional filling factors with excitation gaps for fractionally charged quasiparticles provide all the necessary ingredients for an explanation of the experimentally observed transport features. For the most prominent and best studied of the FQHE states, at $\nu = 1/3$, good quantitative agreement has been reached between theory and experiment. It appears that the FQHE has basically been understood.

Even-Denominator States and Spin

Perhaps the most obvious feature of the hierarchy is the odd-denominator rule. Stemming from the grand Pauli exclusion principle applied to the primitive Laughlin states at $\nu = 1/3, 1/5, \ldots$, this "rule" seemed almost a "law," which it is not. Despite rumblings about possible fractional states at $\nu = 3/4$, 11/4, 5/2, and 9/4, the widespread view was that these "bad actors" would evaporate under closer scrutiny with better samples. To most everyone's surprise and excitement, one of these fractions has survived the critical test: a plateau has recently (*10*) been clearly identified with Hall resistance $R_H = (h/e^2)/(5/2)$. Figure 6 shows solid evidence for the 5/2 state. These data were obtained at the very low temperature of 25 mK, attesting to the fragility of the new liquid state.

This first even-denominator state not only represents an egregious failure of the hierarchical model, but goes to the very root of our picture of the FQHE. Since two levels are completely filled and one is 1/2 filled at $\nu = 5/2$, this state is really a 1/2 state and we are led back to Laughlin's family of $\nu = 1/m$ wave functions. All such wave functions with m an even integer were discarded for not changing sign under interchange of two electrons. While no one seriously doubted this fundamental law of physics, it was clear that fitting $\nu = 5/2$ into the picture required major revision of the standard model. Despite great effort, we still lack a conclusive theoretical understanding of this new surprise from the 2-D electron system.

Suggestions of how one might in principle construct a 1/2 state have been around for several years. The most obvious way was to imagine particles which, under interchange, required their wave function to *not* change sign. Such particles exist in nature, they are called bosons, a helium atom being a notable example. With such particles the Laughlin wave function would be valid only for $m = 2,4, \ldots$, leading to states at filling factors $\nu = 1/2, 1/4. \ldots$ Electrons are not bosons, however, and so this approach is not of much help. Another early suggestion (*11*) for creating even-denominator states turned on a property of electrons which we have so far ignored: the electron spin. Each electron behaves like a tiny bar magnet, which can point either "up" or "down." This intrinsic magnetism of the electron is called spin. In a magnetic field the spin prefers one of the two orientations, which we will call "up." It requires energy, called the Zeeman energy, to force the electron to point "down" against the magnetic field. This energy increases linearly with field. The spin acts to split each Landau level in two, with the Zeeman gap in between. While not nearly as large as the Landau gap, the spin gap doubles the number of integer Hall plateaus. Although not mentioned above, all the odd-integer plateaus ($\nu = 1,3,5 \ldots$) are due to the spin gap, while the even-integer plateaus ($\nu = 2,4,6 \ldots$) are due to the Landau gaps. But how does this help to explain an even-denominator fractional state that occurs in one or the other of the spin sublevels? Halperin (*11*) pointed out that if pairs of electrons with opposite spin could form, one could regard the composite objects as bosons, and even-denominators would follow.

The problem is that it costs energy to flip spins, and at high magnetic fields this was considered prohibitive. In Laughlin's original work the spins were all simply assumed to be "up," an excellent assumption for explaining an effect occurring at enormous magnetic fields. But today's samples are so pure that fractional states can be seen at very low magnetic fields; the $\nu = 5/2$ discovery was made at only 5 T, much lower than the old $\nu = 1/3$ state. At such fields spin flips may be relevant. If reversed spins are important in forming the condensed state at 5/2 filling then the application of a second magnetic field, this one parallel to the 2-D plane, should destroy the state. Why is this so? To good approximation, only the spin-flip energy is affected by such an in-plane field, and it is increased. Loosely speaking, in order for a liquid to form the electrons had to expend some condensation energy on flipping spins. Increasing that expense may eventually prevent the state from forming at all. Recent experiments (*12*) have established just such an effect, lending strong support to the spin reversal hypothesis.

There is no theoretical agreement on the electronic structure of the even-denominator state. One very elegant model (*13*) has been proposed, for which spin reversal is crucial, but it is not clear that it is a viable description of realistic 2-D electron systems (*14*). Some workers (*15*) have even argued that the spins are not reversed at all, which is hard to square with experiment. At present we are far from understanding this obvious inadequacy of the standard model.

One may also fairly ask: If spin is important at the top of the hierarchical pyramid, then what about further below? The answer to this is simply not known yet. Interesting effects have already been observed. Certain fractions, for example $\nu = 8/5$, occur in two distinct hierarchical schemes. In one the 8/5 state has all its spins aligned with the magnetic field, while in the other scheme half the spins are reversed, and the net spin of the state is zero. Which state is lowest in energy? This depends on the magnetic field at which $\nu = 8/5$ occurs. If the spin-reversed variant is lower in energy, then adding a parallel magnetic field will destabilize it, just as with the 5/2 state. Adding a large enough parallel field can result in the spin-aligned 8/5 state becoming lowest in energy. The system thus undergoes a phase transition between the two ground states. The latest experiments (*16*) have uncovered just such phenomena. While most believe the standard model to apply at the highest magnetic fields, at lower fields, where spin becomes important, the subject is far less settled. Still more surprises may be in the offing.

Conclusion

The present picture of the dynamics of 2-D electrons in high magnetic fields is an intricate web of distinct quantum liquid states connected by strange quasiparticle excitations carrying fractional charge. While the dominance of the coulomb interaction was recognized early on, only a small subset of the observed FQHE states is understood in any detail. This has been highlighted by the recent discovery of an even-denominator state and its likely connection to the electron spin. As a consequence, considerable reworking of the hierarchical model is under way.

Essentially all we now know of the FQHE has been determined through one type of experiment: simple electrical conduction. While many other potential probes exist, they are only just beginning to be employed. Optical investigations, microwave absorption studies, tunneling experiments, and thermodynamic measurements will all add significantly to our understanding of the collective states underlying the FQHE.

Having appreciated the dominance of correlations in many-electron systems in high magnetic fields, we expect further manifestations of this phenomenon. In two dimensions, the hierarchy of liquid states should eventually terminate with the electrons freezing into a solid. Much interest surrounds this predicted transition but conclusive experiments have yet to be done. Multilayer 2-D electron systems in which electrons are allowed to interact between planes will allow for novel electron configurations as yet unobserved. Even "old-fashioned" 3-D electron systems are expected to reveal new classes of condensed states. Novel crystal growth techniques are beginning to achieve the dramatic reductions in impurity levels required for the observation of such states. Intense interest has been generated quite recently in one-dimensional electron systems. A fascinating quantization of the resistance, akin to the integer quantum Hall effect, has already been observed (*17*). In retrospect, the diversity of phenomena observed or expected from a system of only electrons seems astonishing. The fractional quantum Hall effect is perhaps only one spectacular example.

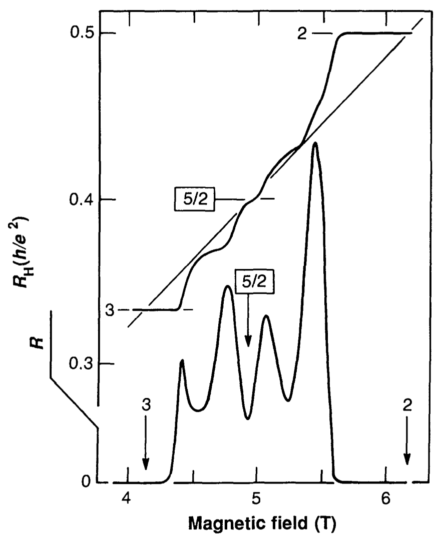

Fig. 6. Observation of a fractional quantum Hall effect at an even-denominator fraction, $\nu = p/q = 5/2$. A plateau is just beginning to form at $R_H = (h/e^2)/(5/2)$ and a strong minimum is seen in the longitudinal resistance, R. This data was obtained at a temperature of only 25 mK. The straight diagonal line gives the classically expected Hall resistance. The nearby integer quantum Hall states at $\nu = 2$ and 3 are also shown.

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