Fractional Statistics: Quantum Possibilities in Two Dimensions

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A standard notion of quantum mechanics is that all particles, elementary or composite, must fall into one of two fundamental categories: fermions or bosons. However, it has recently been discovered that there can be quantum particles which are neither fermions nor bosons. Such particles (anyons) can only occur in two spatial dimensions—yet this does not rule out their existence, for they are found as elementary excitations in confined, quasi–two-dimensional condensed-matter systems and may occur in other systems as well. An overview of the argument for the existence of anyons is presented, along with a discussion of their role in condensed-matter physics.

THERE IS A WELL-ESTABLISHED SET OF IDEAS IN QUANTUM physics which is embodied in the word "statistics." In quantum statistical mechanics, the rules for counting states are drastically modified from the classical rules, in one of two possible ways, depending on the nature of the quantum-mechanical interference arising among identical particles. If the quantum particles are identical bosons, then the amplitudes for configurations which differ only by a permutation of two particles interfere constructively—that is, with a relative phase which is $+1$. In contrast, the interference is destructive—the relative phase is -1—for identical fermions. The effect of this difference is not small: any number of bosons may occupy a single quantum state, while the occupation numbers for fermions are restricted to the values 0 or 1 (the Pauli exclusion principle). At low temperatures, these simple counting rules can dominate the physics. A gas of fermions (for example, electrons in a metal) will occupy states of very high kinetic energy—up to the Fermi energy—even at zero temperature, simply because of the Pauli principle. A Bose gas (such as liquid helium) can, by contrast, "condense" at low temperatures into a state in which a large fraction (up to 100%) of the particles are in the same quantum state and are phase-coherent: a superfluid. All the low-temperature properties of a system of many identical particles are thus controlled by the simple difference between $+1$ and -1. Because of the tight relationship between counting rules for the occupation of states (in statistical mechanics) and the relative exchange phase associated with exchanging two particles (in quantum mechanics), the word "statistics" has come to refer to both.

The "amplitude for a configuration" (for example, N particles at $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\}$) is of course the wavefunction $\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$. Let us call the amplitude which differs only by a single permutation $\psi' \equiv \psi(\mathbf{r}_2, \mathbf{r}_1, \ldots, \mathbf{r}_N)$. The identity of the particles means that the probability density is unaltered by the exchange:

$$|\psi'| = |\psi|^2 \tag{1}$$

or

$$\psi' = p\psi \tag{2}$$

where p is a simple phase $\exp i\theta$. A second exchange gives $\psi'' \equiv p^2\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$. Requiring the wavefunction ψ to be single-valued, we then obtain $\psi'' = \psi$, or $p^2 = 1$. That is, the permutation operator has only two possible eigenvalues

$$p = \pm 1 \tag{3}$$

Thus, using extremely simple arguments, we find that only two kinds of wavefunctions are possible for identical particles, and nature has (apparently) obliged by presenting us with the corresponding kinds of particles: bosons ($p = +1$) and fermions ($p = -1$). It is naturally tempting to believe that this coincidence—two kinds of permutation symmetry for wavefunctions, and two kinds of particles found in nature—is explanatory, that is, that the former fact explains the latter. The chief purpose of this article is to demonstrate that this property of wavefunctions does not in itself determine the possible phases which may arise upon physical exchange of identical particles; that a more careful look at the question will again give us only bosons and fermions in three dimensions, but a continuum of possibilities ("fractional statistics") in two dimensions; and, finally, that such other possibilities are worth serious consideration, as they can arise in some condensed-matter systems which are effectively two dimensional. These ideas were not originated by us, being roughly 10 years old (*1*) [and the groundwork was laid still earlier, in 1971 (*2*)]. We are simply a small part of a rapidly growing group of physicists who are currently trying to understand the physics of this new kind of two-dimensional particle.

Paths, Topology, and Statistics

We are used to thinking that the permutation eigenvalue p controls the main physical properties of a system. It turns out however, that the quantity which is crucial to the physics is not p but rather the phase which arises from the adiabatic transport of two particles [or, equivalently, in the sense of Berry (*3*), of the state ψ] along a path which gives an actual physical exchange. This latter phase (let us call it η) depends upon both the wavefunction and the Hamiltonian, as we shall show below. Since η does not necessarily

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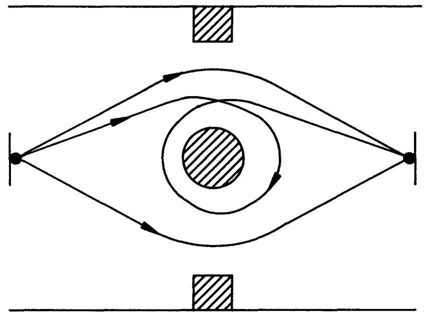


Fig. 1. An idealized sketch of an Aharonov-Bohm experiment. Particles are emitted at the left and detected at the right. There is a magnetic flux Φ in the impenetrable solenoid at the center. Paths which differ in relative winding number about the solenoid suffer a phase shift which depends on the enclosed flux; three such paths are shown.

equal p (although, for bosons and fermions, the two can easily be the same), one must choose which one is meant by the term "statistics." We shall choose it to mean η. Before justifying this claim ($\eta \neq p$) explicitly, let us first examine the possible constraints on η by considering the kinds of paths (*1, 2, 4*) which may arise in a space with identical particles—in other words, we seek the analog of Eq. 3 for η.

The quantum amplitude associated with a path is called a propagator (*5*). Let $K(\boldsymbol{R}, \boldsymbol{R}')$ be the amplitude for propagation from $\boldsymbol{R}$ to $\boldsymbol{R}'$, where $\boldsymbol{R}$ stands for a whole configuration $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\}$. Then K can be written as a "path integral"—that is, as the sum of partial amplitudes $k_\omega(\boldsymbol{R}, \boldsymbol{R}')$, one for each possible path ω from $\boldsymbol{R}$ to $\boldsymbol{R}'$:

$$K(\boldsymbol{R}, \boldsymbol{R}') = \sum_{\omega} \chi(\omega) k_\omega(\boldsymbol{R}, \boldsymbol{R}') \tag{4}$$

The path amplitude k for a given path ω is determined by the Hamiltonian H, or, equivalently, the action S, according to $k_\omega =$ esp $iS[\omega]$. The weights χ are then needed to account for any boundary conditions (such as minus signs due to Fermi statistics) which are not taken care of in H. We deduce several things (*2, 6*) about the weights $\chi(\omega)$:

1) Paths which can be continuously deformed into one another (that is, without cutting—although stretching or shrinking is allowed) must have the same weight χ. Clearly the differences in k for two such paths must arise from the action S and not the boundary conditions. This constraint allows us to group the paths in space into classes α, and to rewrite Eq. 4 as

$$K(\boldsymbol{R}, \boldsymbol{R}') = \sum_{\alpha} \chi(\alpha) \sum_{\omega \in \alpha} k_\omega(\boldsymbol{R}, \boldsymbol{R}') \tag{5}$$

(We note that if there is more than one class in the space—that is, if there are paths with common end points which cannot be deformed into one another—then the space is "multiply connected." Our whole discussion is only interesting if this is the case. For concreteness, one might imagine particles confined to a ring or punctured disc. The different classes are then the paths with different winding numbers about the excluded region.)

2) We can choose $\boldsymbol{R}$, $\boldsymbol{R}'$, and S in Eq. 5 so as to make the contributions of all classes of paths but one (say, $\hat{\alpha}$) in the right-hand side arbitrarily small. Then, taking the absolute value of both sides tells us that $\chi(\hat{\alpha})$ is unimodular, that is, a simple phase. This clearly holds for any class $\hat{\alpha}$.

3) Finally, we restrict ourselves to closed paths ($\boldsymbol{R} = \boldsymbol{R}'$). This allows us to imagine any path in a given class γ as a composition of paths in different classes (say, α and β). (For example, any path with winding number $n = 3$ around a ring may be seen as the composition of an $n = 2$ path and a path with $n = 1$.) Since we can do this for any path in γ (by continuous deformation) without changing the classes of the "composing" paths, the relation holds for the classes themselves. Schematically, we express this composition of classes as $\gamma = \alpha\beta$. The corresponding result for the weights in the path integral is clearly

$$\chi(\gamma) = \chi(\alpha)\chi(\beta) \tag{6}$$

[For closed paths, the classes are called homotopy classes, and their group structure under composition on a manifold is the fundamental homotopy group (Π_1) of the manifold (*7*). Equation 6 then says that the weights χ form a scalar representation of the group Π_1.]

As a counterpoint to these mathematical abstractions we mention the simple, idealized Aharonov-Bohm effect (*8, 9*), in which charged particles move in a field-free region surrounding an impenetrable core containing magnetic flux (Fig. 1). Because the particles can never reach the field region, the magnetic flux has no classical effect (such as a Lorentz force) whatsoever. However, there is a purely quantum phase which appears from the vector potential $\mathbf{A}$ and which can cause a shift of the two-slit diffraction fringes in the experiment. The vector potential induces a topological phase $\exp(in\Delta)$ where

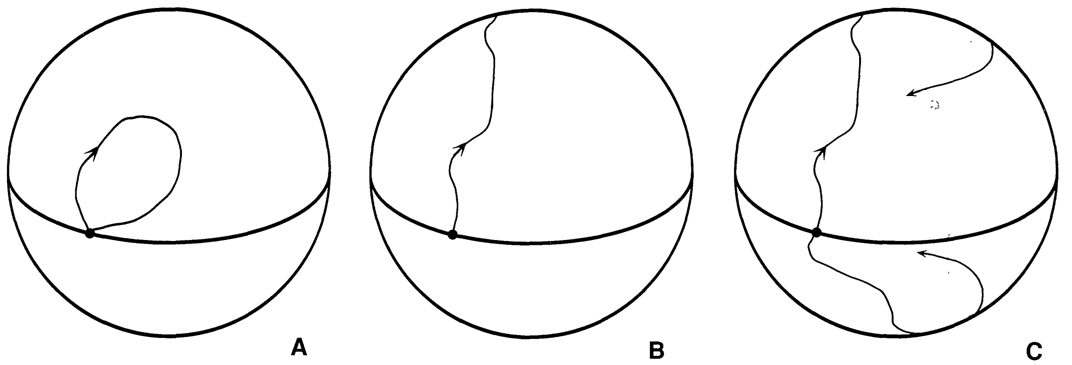
$$n\Delta = \frac{e}{\hbar c} \oint_\Gamma d\mathbf{r} \cdot \mathbf{A} = 2\pi n \frac{\Phi}{\Phi_0} \tag{7}$$

n is the winding number of the path Γ around the flux Φ, and $\Phi_0 \equiv hc/e$ is the flux quantum. We then have two choices for assigning the weights χ to paths with different winding number n. We can include an appropriate vector potential in H and let all the χ_n be 1. The alternative is to use a free-particle Hamiltonian, in which case the flux appears in the weights χ_n as a boundary condition: $\chi_n = \exp in\Delta$. We see that this latter choice obeys the composition rule (*4*) (the former does so trivially): $\chi_n \chi_m = \exp i(n + m)\Delta = \chi_{\{l=n+m\}}$. Either choice yields the correct physics (*8, 10*).

Now we address the question of statistics, that is, the exchange of identical particles. We want a formalism that does not artificially place labels on indistinguishable particles. Hence, we define a configuration space C^d as a set of points in Nd dimensions, representing the positions of N particles in d spatial dimensions—but, in C^d those points which are the same to within a permutation of the particles are identified as one point. Exchange paths in this space are thus closed paths. If we further exclude from this space all points of double occupancy (accomplished physically by a hard-core repulsion), then it turns out C^d is in fact multiply connected. We want to examine the set of closed paths for this space. If we can determine the composition laws of these paths, then Eq. 6 will give us the constraints on the exchange phase, which is called η above. For simplicity we will treat only two particles, and restrict our attention to their relative coordinate $\mathbf{r}$—recalling that, in C^d, $\mathbf{r} = -\mathbf{r}$.

We first take $d = 3$. Without resorting to homotopy theory, we can classify the closed paths in C^3 using simple pictures, as in Fig. 2. Given the hard-core constraint ($\mathbf{r} = \vec{0}$ is forbidden), we can hold $|\mathbf{r}|$ fixed without loss of generality. We thus imagine two particles, moving in three dimensions relative to one another at a fixed separation (so that the locus of $\mathbf{r}$ is the surface of a sphere). The resulting manifold (P_2—a subspace or "shell" extracted from C^3 for purposes of visualization) is a spherical surface, with opposite points identified. Let us now use P_2 to characterize the kinds of closed paths—which constrain η—in the configuration space C^3. Figure 2a shows a path on P_2 in the trivial class (no exchange), to which we assign the weight $\chi_0 = 1$. All paths in this class may be continuously shrunk to a point. In Fig. 2b we see a closed path (remember, $\mathbf{r} = -\mathbf{r}$) which cannot be deformed to a point and thus represents a class distinct from the trivial class. This path represents a single exchange of the two particles and its class receives the weight η. Finally, in Fig. 2c we show a path which represents two exchanges

Fig. 2. Paths showing relative motion of two identical particles, by tracking their relative coordinate **r**. (**A**) A path which does not involve an exchange. This path can be continuously deformed to a point. (**B**) A path involving a single exchange ($\mathbf{r} \to -\mathbf{r}$). This path is a closed loop since $\mathbf{r} = -\mathbf{r}$; however, it cannot be deformed to a point without cutting. (**C**) A path involving two consecutive exchanges. The attached lines show how to deform this closed path to a point, by dragging it around the sphere.



(weight η^2). It is clear from the figure that this path may be continuously deformed (that is, without cutting) into the trivial path (Fig. 2a). It thus falls in the same homotopy class and takes the same weight in the propagator: $\chi[(c)] = \eta^2 = \chi_0 = 1$. We find:

In three dimensions, two exchanges are topologically equivalent to zero exchanges. Thus the phase assigned to two exchanges must be that assigned to the case of no exchange.

In other words, by a considerably more involved argument, for the three-dimensional case we obtain the same result as Eq. 3: $\eta^2 = 1$ and hence there are only bosons and fermions.

The reward for our investment of effort in this rather formal language comes when we consider the two-dimensional case. The appropriate manifold for this case is a circle with opposite points identified: P_1, rather than P_2 (Fig. 3). Again it is clear that a single exchange cannot be deformed to the trivial path. It is also clear, however, that neither can the two-exchange path (once around P_1) be so deformed, nor in fact can the path representing n exchanges, for any n. This means that the number of classes is infinite, rather than two. This also means that there is no constraint on the value of η^n, and thus on η, for two-dimensional particles:

In two dimensions, two exchanges are not topologically equivalent to zero exchanges, nor are n exchanges, for $n \neq 0$. Thus it is possible to consistently assign any value to the phase due to exchange.

The technical name for the group structure of the paths in C^2 [the first homotopy group, $\Pi_1(C^2)$] is the braid group (*4, 11*). We mention the name because it is apt, and picturesque. If we imagine world lines for particles moving in 2 (space) + 1 (time) dimensions (Fig. 4), we see that, given the hard-core constraint, the world lines cannot cross. Thus one can, in principle, distinguish crossing "in front" from crossing "behind"—that is, clockwise from counterclockwise. Put more simply, in two dimensions it is possible for the statistics of particles to possess a detectable (right- or left-) "handedness." Furthermore, a third world line can be braided into the two shown in a nontrivial way—that is, in a braid that cannot be undone (Fig. 4b). We can also see this algebraically: if an exchange path of two particles happens to enclose a third, the appropriate phase is $\eta \cdot \eta^2$. The latter factor results from a net encirclement of the third particle (which is equivalent to two exchanges) and is nontrivial when $\eta^2 \neq 1$. This fact makes the many-particle problem, for arbitrary statistics in two dimensions, considerably less trivial than the many-boson or many-fermion problem: the phase η due to exchange of two identical particles in two dimensions depends in principle on the positions of all the other particles.

From the above arguments we see why consideration of simple permutation symmetries of wavefunctions does not suffice in two dimensions. A given permutation has no information on the path by which the permutation is accomplished. We have shown that the topology of space for identical particles in three dimensions allows for only two classes of exchange paths and thus only two values of the exchange phase η. These values, being ± 1, can be treated by a formalism which only tracks the net permutations of the particles. The two-dimensional case, however, allows a continuum of values for η, because there are an infinite number of classes of exchange paths. For this case, one clearly needs more information than is contained in the simple two-valued permutation. We suggest, then, that it is a fortunate historical accident that, on the one hand, there are only two kinds of wavefunctions which are eigenstates of permutation, while, on the other hand, our world is apparently (at least) three dimensional, so that we have only been presented with particles (bosons and fermions) whose exchange properties are path-independent.

There is another way of viewing the paths in Fig. 2 which is illuminating. The single-exchange path (b) can be smoothly deformed into its time reverse by simple rotation about the "north pole" (Fig. 5). In quantum mechanics, time reversal T is equivalent to taking the complex conjugate of the amplitude. Since the path (b) and its time reverse are in the same homotopy class, we have that $\eta^* = \eta$, which again gives $\eta = \pm 1$. For the two-dimensional (2D) case, however, it is clear that there is no way to deform a path into its time reverse: these are paths with opposite winding numbers on P_1. This again tells us that $\eta[2D]$ is not constrained to the values ± 1. We also see that two-dimensional particles which are neither fermions nor bosons do not obey time reversal symmetry. This is in fact an outstanding signature of particles with nonstandard statistics, which should be an aid both in thinking about their properties and in detecting their existence.

At this point, we have summarized results which were (mostly) in print in 1977. The application of homotopy theory to the composition rules for the weights χ was accomplished by Laidlaw and DeWitt (*2*) in 1971, but they did not pursue the implications of their result for the two-dimensional case. The possibility of a continuum of values for the exchange phase in two dimensions was (to our knowledge) first treated in 1977 by Leinaas and Myrheim (*1*) who were apparently unaware of the work of Laidlaw and DeWitt. Finally, we note that, if the hard-core constraint is not imposed, then C^d is simply connected for any d: all paths are deformable to the trivial path, and only bosons are possible.

Dyons and Anyons

We can illustrate the ideas of the previous section with a simple physical picture, which will bring us up to 1982 and allow us to introduce both the name and a model for these proposed particles. Again we begin with the three-dimensional case and consider a particle with charge e, tightly bound (somehow) to a magnetic monopole (*12*) of magnetic charge g. The resulting composite is called a dyon (*13*). The charge g is quantized (in terms of e) by a number of consistency conditions (*13*).

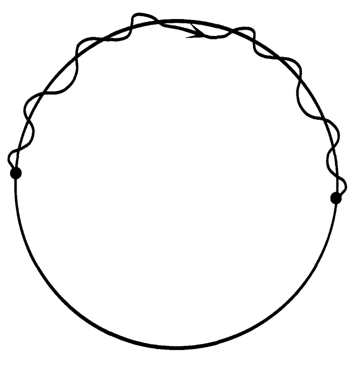


Fig. 3. The two-dimensional version of Fig. 2. Shown is a closed path involving a single exchange in two dimensions. This path cannot be deformed to a point—but neither can a path involving two (or more) exchanges.

We now consider adiabatic exchange of two identical dyons (each of which has a single magnetic charge g), keeping track of the phase arising from the relative motion of charge and flux. The result (*13*) is that, if the charged particles are bosons, the net phase upon exchange is -1, arising solely from the electromagnetic coupling. Similarly, if the charges are fermions, the dyons acquire a phase of $+1$ upon exchange. This is our first example of a case in which a wavefunction can be of a given permutation symmetry p, while the exchange phase η can be completely altered from p, as determined by the electromagnetic coupling in H. The quantization of g prevents any result more exotic than transmutation from Bose to Fermi, or the reverse, in three dimensions. This, however, should not be surprising in the light of the above topological argument.

Now [following Wilczek (*14*)] we consider the two-dimensional analog of the dyon. The charge e is unchanged, but the analog of the monopole is an infinitely thin flux tube piercing the charge (the construction of such an object is graphically illustrated in Fig. 6). In two dimensions, one can consistently allow the flux to take on any value, just as in the Aharonov-Bohm effect. Thus, the adiabatic exchange of two identical two-dimensional charge-flux composites can in principle give any phase. Wilczek therefore named these objects anyons—and the name has since been generalized to any two-dimensional object with nonstandard statistics.

The anyon picture allows us to present a simple model which again demonstrates the independence of η from p—and the fact that the physics is determined by the former. We consider two bosons (charge e) bound to the ends of a rigid rod which is itself confined to a plane. Neglecting the center-of-mass motion, the problem has one degree of freedom: the relative angle φ. The two-body wavefunction ψ must be symmetric under exchange, since the particles are bosons; but exchange simply adds π to ϕ:

$$\psi(\varphi) = +\psi(\varphi + \pi) \tag{8}$$

The Hamiltonian, eigenfunctions, and eigenvalues are

$$H = \frac{L^2}{2I} = \frac{\hbar^2}{2I}\left(-i\frac{\partial}{\partial\varphi}\right)^2 \tag{9}$$

$$\psi(\varphi) = e^{im\varphi}\,;\, E_m = \frac{\hbar^2 m^2}{2I} \tag{10}$$

where L is the angular momentum and I is the moment of inertia. The symmetry requirement then restricts the boson spectrum to even values of m and the ground-state energy is zero.

Now we bind flux tubes to the bosons. Note that the particles never contact the flux, responding only to the vector potential. (We assume throughout that particles do not see their own flux.) The Hamiltonian in the presence of the flux tubes is

$$H' = \frac{\hbar^2}{2I}\,(-i\frac{\partial}{\partial\varphi} + \theta/\pi)^2 \tag{11}$$

where $\theta = 2\pi\Phi/\Phi_0$ controls the phase under adiabatic exchange $\eta = \exp(i\theta)$. The new energy eigenvalues are

$$E'_m = \frac{\hbar^2}{2I}\,(m + \theta/\pi)^2 \tag{12}$$

For a suitable choice of flux, we can have $\theta/\pi = 1$, making the energy spectrum of bosons, whose wavefunction is symmetric (m even), precisely equivalent to that of fermions (without attached flux). One can in fact show that for $\theta = \pi$, every observable in the system has the value identical to what it would have been if the particles had been fermions (without flux tubes). For instance, the angular "Fermi" velocity in the ground state, $(\hbar/I)(\theta/\pi)$, is nonzero (even though $m = 0!$)—again giving (for $\theta = \pm\pi$) the value appropriate to fermions. For nonintegral values of (θ/π), the particles are effectively anyons: their statistics is "fractional." (Here we see that the term "fractional" statistics refers to the "statistics angle" θ being a fractional multiple of π.) This example, admittedly simple and artificial, nevertheless amply underscores our claim: p (permutation symmetry of ψ) does not necessarily equal η, the exchange phase—and the latter determines the physics.

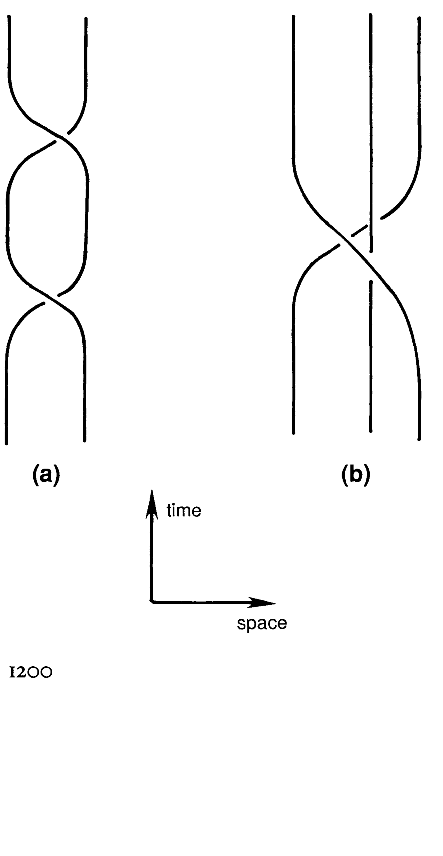


Fig. 4. The braiding of world lines which cannot pass through one another in 2 + 1 dimensions. (**a**) Shows that two exchanges cannot be "undone," (**b**) shows that a third "passive bystander" is nontrivially involved in general.

What About the Real World?

An important component of our thesis remains to be addressed. How can we expect the possibility of fractional statistics to be realized in nature? Our world is three dimensional, and in it we find only bosons and fermions. We know that many condensed-matter systems can be rendered effectively two dimensional, but is it not the case that such systems will still be inhabited only by the fundamental three-dimensional particles, that is, by bosons and fermions?

We can find a hint of the answer from the anyon picture. If it were possible to truly bind a charge e to a fraction of a flux quantum (keeping the resulting object confined to two dimensions), then fractional statistics would be physically realized. It turns out that nature has accomplished this task, with a slight twist: by binding a fractional charge to a single flux quantum—or, more accurately, to a 2π vortex.

Let us elaborate on this claim (*15*). In condensed-matter systems called semiconductor heterostructures, it is possible to create a two-dimensional electron fluid which, when subjected to large magnetic fields and low temperatures, exhibits the quantum Hall effect, and

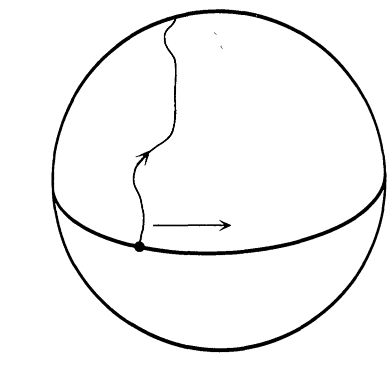


Fig. 5. How to deform the three dimensional exchange path (Fig. 2B) into its time reverse, without cutting, by simply rotating it about the "north pole" as shown—keeping the "equatorial" points diametrically opposite and leaving the polar point fixed.

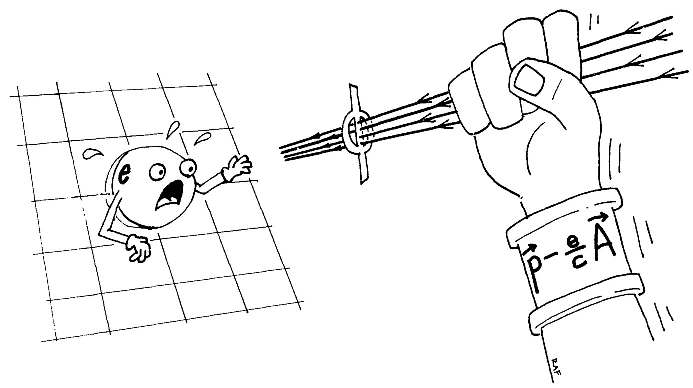


Fig. 6. The piercing of a charge (which is confined to two dimensions) with a flux tube. The resulting composite can have fractional statistics. To date, experimentalists have not succeeded in performing this operation; however, nature has been (as always) more clever. [Copyright 1989 World Scientific. Reprinted from Shapere and Wilczek (*26*) with permission]

particularly, the fractional quantum Hall effect (FQHE). [Since it is not our main purpose to treat the physics of the FQHE, for which there is a large literature available (*16, 17*), we offer here a highly condensed and heuristic discussion. The actual calculation of fractional statistics, by adiabatic transport of a good variational state, was performed by Arovas *et al.* (*15*).] For our purposes it suffices to state that the FQHE fluid, at certain "magic" ratios ν of electron density to flux density, is incompressible—there is a finite energy needed for deviations in the electron density from the special value. We call these special states ψ_m (where m is an odd integer). Such a state is characterized by the existence of precisely m flux quanta for each electron, giving $\nu = 1/m$.

It turns out that there exist excitations in this fluid which are vortices (*18–20*). A simple example of a variational wave function for a single vortex is Laughlin's quasihole state (*18*)

$$\Psi_v = \prod_{j=1}^{N} (z_j - z_v)\Psi_m \tag{13}$$

where $z_j = x_j + iy_j$ is a complex representation of the particle coordinates, z_v is the vortex position, and Ψ_m is the $\nu = 1/m$ ground state. It is clear from Eq. 13 that there is a vortex at z_v, in the sense that adiabatic transport of any single particle (z_j) in a closed path encircling z_v gives a phase winding of 2π; the phase winding is manifest in the analytic zeros of Ψ_v. We see that this excitation acts very much like a flux tube with an integer amount of flux. Now we ask, what "charge" is associated with this "flux"? The counting is straightforward: there is one vortex for each flux quantum (excess or deficient) relative to the preferred value $\nu = 1/m$. The incompressible fluid then excludes from itself precisely $\pm 1/m$ of an electron per vortex in order to maintain the commensuration $\nu = 1/m$ everywhere away from the vortices. The resulting composite—a fraction of an electron ("charge"), bound to an integral vortex ("flux")—has fractional statistics (*15*)

$$\theta/\pi = 1/m \tag{14}$$

This, briefly, is how nature has made an anyon: by binding a well-defined, fractional charge to an integral vortex in a two-dimensional fluid. We should emphasize that the charge is bound to the vortex—the zero of the wavefunction—and not to a physical quantum of magnetic flux.

Several suggestions have been made for other candidates for fractional statistics—for example, vortices in superfluid He films (*21*). Laughlin (*22*) and Wen, Wilczek, and Zee (*23*) have argued on rather general and compelling grounds that two-dimensional, frustrated spin systems should contain excitations carrying fractional (1/2) spin (instead of charge) excitations which obey fractional statistics. Laughlin has found a somewhat artificial, but nevertheless intriguing, model (*24*) for which the ground state has precisely these properties (because it is equivalent to his FQHE ground state for bosons at $\nu = 1/2$). Note that the broken time reversal (T) symmetry of the fractional statistics quasiparticles in the FQHE case is easily understood, since the symmetry is already broken by the magnetic field. In the case of frustrated spin systems, the appearance of excitations with fractional statistics implies that T symmetry is broken spontaneously (which is not unusual for magnetic systems). Such a system could then assume either chirality (right- or left-handed).

Let us briefly review the ideas presented thus far. We have presented an argument for the existence in two dimensions of objects which are neither fermions nor bosons, but rather entities which give rise to a complex phase upon exchange. The argument is based on an appeal, not to the properties of wavefunctions, but to the topology of space and the identity of quantum particles. We have shown that such objects (anyons) must in fact have a handedness or chirality, that their properties do not respect time reversal symmetry, and that the many-particle problem is significantly more complex than that for bosons or fermions. We can name one system (the FQHE) for which there is strong evidence of well-defined quasiparticles with fractional charge and fractional statistics. Given our belief in the above reasoning, and in the correctness of the FQHE picture, we see no reason to expect the number of systems in which fractional statistics may be physically realized in the laboratory to remain fixed at one. Thus we feel that systems of anyons merit attention and study because they are possible—quantum mechanically and physically.

Quantum Mechanics with Fractional Statistics

Let us then take anyons seriously and ask, what might be the properties of an "ideal gas" of such particles? We might ask, for example, what the momentum distribution function is. Is it in some sense intermediate between Bose and Fermi? The answers to such questions are difficult to obtain, due to the complexity of the "many-anyon" problem alluded to above. This problem may be attacked in basically one of two ways—just as we did for the Aharonov-Bohm problem. One option is to treat the statistical phases as arising from a fictitious flux which is attached to bosons (or fermions), and which thus appears in a Hamiltonian for bosons (or fermions) which are interacting—via a long-ranged, vector potential. Alternatively, the anyons may be treated as "noninteracting" if their statistics is treated as a boundary condition to be imposed on eigenfunctions of a free-particle Hamiltonian. This is the standard approach for the ideal Bose and Fermi gases. The catch is, for anyons, the exchange phase is path-dependent. Thus a wavefunction for noninteracting anyons, obeying the proper boundary conditions, must be multivalued; it may be written in the form (*4, 25*)

$$\psi(Z) = \prod_{i<j} \frac{(z_i - z_j)^{\theta/\pi}}{|z_i - z_j|^{\theta/\pi}} f_s(Z, Z^*) \tag{15}$$

Here $Z = \{z_1, z_2, \ldots, z_N\}$ is the configuration of N particles in terms of the complex coordinates $z_i = x_i + iy_i$. The function f_s is completely symmetric in the $\{z_i\}$, and the proper phases are accounted for by the prefactor of f_s, which is a pure phase and is multivalued for $\theta/\pi \neq$ integer. The parameter θ is again the statistics angle: $\eta = \exp i\,\theta$. While Eq. 15 is very compact and gives the proper phases, it has had limited use because it is difficult to find (single-valued) functions f_s for which $\psi(Z)$ satisfies the free-particle Hamiltonian. There is, however, a partial solution to a three-body problem in the form of Eq. 15 by Wu (*4*).

The two-body problem can be solved exactly. This information can then be used to compute thermodynamic quantities which depend only on two-body properties. Examples are the second virial coefficient of an ideal anyon gas (*25–27*) and the specific heat of a gas of dimers (*25*) whose spectrum is given above in Eq. 12.

The original anyon model (charge/flux composite) has proven to be the most useful approach so far for treating many anyons. In this picture, anyons are charged bosons or fermions with attached flux. The charge and flux are both in a sense fictitious. That is, they can be assumed to have no effect other than to give rise to the appropriate phases for relative motion. The particles are bosons or fermions in the sense that $p = \pm 1$, that is, their wavefunctions have good permutation symmetry. The price paid, as noted above, is that their true statistics η is placed in the Hamiltonian—the particles interact with a long-ranged vector potential. As we have tried to show before with a simple example, this transformation (that is, placing all or part of the statistical interaction η in the Hamiltonian) is exact: for example, bosons with $\theta = \pi$ really are fermions. This model allows us to work with well-known tools: single-valued wavefunctions, well-defined commutation relations, and so on, but the statistical interaction clearly is not in any sense "weak."

One reason the "original anyon" model has been useful for theorists is that, once the particle statistics is treated as an interaction, one can imagine treating this interaction at the mean-field level; that is, there can be a mean-field theory of statistics (*26, 28, 29*). If we take one anyon as a charge with bound flux and ask how he sees his fellows on average, the answer is: as a uniform magnetic field. The strength of this field is, of course, determined by the density and statistics (η!) of the particles. If the wavefunction is symmetric (the anyons are interacting bosons), then the bosons must be given a hard-core repulsion to enforce the prohibition against double occupancy. Thus the simplest approach to using the mean-field theory of statistics is to let the wavefunction be appropriate to fermions—which automatically enforces the constraint—and then to correct η from (-1) with attached flux. This model, at the mean-field level, is exactly soluble: it is noninteracting two-dimensional electrons in a uniform magnetic field, whose states are Landau levels (*16*).

The notion of a mean-field theory of statistics is more than just a useful calculational technique. It is a novel source of ideas and insights. In the last paragraph, it was implicitly assumed that the fictitious or "statistical" flux, when smeared out or averaged, could in fact be modeled, not by a uniform fictitious flux, but by a true magnetic field. This is, in fact, not a small step: the former flux, besides being highly nonuniform and mobile, has no classical effect whatever, while the latter produces a classical Lorentz force on the particles and is uniform. Nevertheless, the idea is very interesting. Let us suppose that the mean-field theory is not a bad approximation. Then an experimentalist can, after confining a batch of electrons to two dimensions in the laboratory, effectively alter their statistics with a magnet!

We have, of course, overstated the case, but there is some truth in the idea. For instance, for the FQHE fluid (m flux quanta per electron, m an odd integer), the electrons are, in the sense of mean-field theory, bosons (in zero net field) (*30, 31*). Does it make sense to think of the FQHE fluid as a two-dimensional Bose superfluid? The answer seems to be, yes, with some caution (*20, 32*). The FQHE fluid can support a dissipationless flow of current. Its excitation spectrum is similar to that of superfluid He in that there is a "roton" minimum and its localized excitations are vortices. Furthermore, it is possible to define a kind of one-body off-diagonal long-range order (*31*) in the density matrix analogous to that for the Bose superfluid. The differences between the two fluids are also large (*20*), which is not surprising: the FQHE fluid is incompressible, the energy for a single vortex is finite, the vortices are charged, and, of course, it exhibits a Hall effect. Nevertheless, the FQHE is a good example of the utility and interest of the ideas implicit in the mean-field theory of statistics: that a magnetic field can, in some sense, renormalize the statistics of particles in two dimensions.

Thus, the idea of a mean-field theory of statistics—which is truly novel, and not obviously reasonable—has some appeal, and some limitations. How might it be tested? The FQHE itself is a partial test of mean-field theory, as we have just described; but there are better tests. Laughlin (*28*) constructed a variational wavefunction in the form of Eq. 15, with the important difference that the function f is antisymmetric—it is the mean-field fermion solution. The prefactor then gives the correct statistics η—for instance, for $\theta = \pi$ the fermions become bosons. Fetter, Hanna, and Laughlin improved upon the mean-field solution in another way, using perturbation theory (*29*) to include fluctuations about the mean field. In each case, the results obtained for "effective bosons" were physically reasonable—for example, a logarithmically diverging vortex energy (*28*), and a linearly dispersing phonon mode at small wavevector—restoring the compressibility (*29*).

We have also been able to test the mean-field theory. We have numerically obtained exact solutions for a few ($\sim$8) anyons on finite lattices (*30*) and compared them to the mean-field theory results for the same problem. We placed the statistics (η) in H—that is, the anyons were modeled as bosons with attached flux. Thus, we could directly compare two (formally) bosonic wavefunctions: ψ_{ex} for bosons with attached flux (which are exactly anyons), and ψ_{mf} for bosons in a uniform magnetic field. The direct comparison of the two is then the overlap $M = |\langle\psi_{ex}|\psi_{mf}\rangle|$. Figure 7 shows a contour plot of the magnitude M as a function of the statistical flux attached to the bosons (horizontal axis) in ψ_{ex}, and the uniform flux in ψ_{mf} (vertical axis). According to mean-field theory there should be a "ridge" of large overlap M, starting from the origin (where both fluxes are zero!). Its slope in the figure should be numerically equal to the particle density—which is the scale factor for converting the two fluxes. These predictions are generally well confirmed in Fig. 7, and in the other cases which we tested. Our results for finite systems thus give some credence to the notion that a mean-field theory of statistics, while admittedly an approximation, may not be a bad approximation for treating statistics in two dimensions.

Finally, we comment briefly on the two-dimensional analog of the spin-statistics connection. The spin s of a quantum particle determines the phase δ which is associated with a 2π rotation, according to $\delta = \exp i2\pi s$. The statistics η is the phase due to exchange of two such objects: $\eta = \exp i\theta$. The spin-statistics theorem says that

$$\delta = \eta \tag{16}$$

We note that it is at least possible for such a connection to hold in two dimensions, for fractional statistics, because the two-dimensional rotation group $SO(2)$ is infinitely connected (as is C^2) and so allows for any eigenvalue s for the spin (*33*). This means, less esoterically, that a closed path in the continuous group $SO(2)$, representing a rotation by $n2\pi$, cannot be deformed to the trivial path (no rotation) for any n. In three dimensions, we had to become used to the idea that a 2π rotation is not equivalent to no rotation. In two dimensions, the same holds for any multiple of 2π.

Thus, from the topological point of view, the spin-statistics connection is plausible. It requires a mapping of paths in one space $[SO(2)]$—representing rotations—to paths in another space with similar topological properties (C^2)—representing exchanges. The three-dimensional version can be seen in the same light since the two spaces in this case—$SO(3)$ and C^3—are doubly connected. We mention three approaches to the spin-statistics connection, in order of increasing formality, which apply to fractional statistics: a simple stunt with a belt, performed by Feynman (*34*), a more respectable but still accessible presentation by MacKenzie and Wilczek (*33*), and a highly formal derivation by Fröhlich and Marchetti (*35*). Wilczek's original papers on anyons (*14*) also emphasized that they obeyed the generalized spin-statistics connection—as do their three-dimensional analog, dyons. It should be noted that the spin (intrinsic angular momentum) is associated with allowing the charges to see their own flux.

Ground States for Many Anyons

In spite of a paucity of exact solutions for the thermodynamic limit (*36*), there is a growing consensus among workers in this field that certain many-body ground states which occur for fermions or bosons should have analogs in the anyon case. One class of states are those of the quantum Hall effect. The history of this idea, briefly, is as follows. Experiments have shown the characteristic signatures of the FQHE for Landau level filling fractions $\nu \neq 1/m$, in addition to the (stronger) signs at $\nu = 1/m$. In 1983, Haldane (*37*) proposed a remarkable picture for these unexplained fractions: that the quasiparticles, arising in increasing numbers as ν is tuned away from $1/m$, could themselves "condense" into a QHE liquid when an analogous commensuration is obtained. Haldane's insight, which was key to this picture, was that the quasiparticles, which are vortices, acquire phase shifts when they move with respect to the particles (electrons), just as do particles moving with respect to vortices. Hence, for the quasiparticles, the "parent" QHE fluid plays the role of the uniform magnetic field. The "analogous commensuration" is thus of quasiparticles to particles (with care to avoid double counting), just as the commensuration in the parent fluid is of particles to flux. Since the picture obtained is essentially a QHE within a QHE (with possible recursion), these states are called "hierarchy" states. This picture of the hierarchy states as a QHE of fractional statistics quasiparticles, embedded in a parent QHE fluid, is generally accepted, although by no means proven.

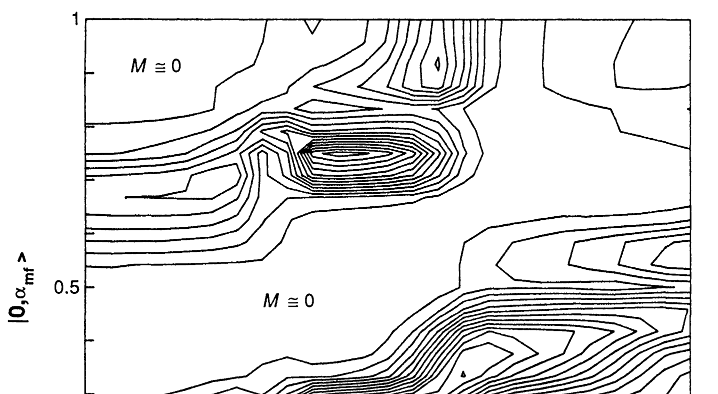


Fig. 7. Contour plot of the overlap M of exact anyon ground states and mean-field ground states. The exact anyon states are obtained by attaching "flux" (α_s) to bosons; the mean-field states, by spreading a uniform flux (α_{mf}) throughout. According to mean-field theory, the overlap of the two (M) should be large along a ridge of slope 1/2 ($\Delta\alpha_{mf}/\Delta\alpha_s$) extending from the origin. This ridge is visible in the figure; the overlap is typically ~0.7 along the ridge. [Copyright 1989 American Institute of Physics. Reprinted from (*30*) with permission]

Our numerical work with small systems tends to confirm this picture. We use the fact, as shown by Girvin and MacDonald (*31*), that it is possible to define a type of one-body off-diagonal long-range order (ODLRO) which characterizes the QHE fluid, just as the Bose superfluid is characterized by a simple one-body ODLRO. Suppose ψ_b is a Bose wavefunction showing ODLRO, and suppose further that some other particles with statistics $\eta = \exp i\theta$ have a wavefunction (in the "noninteracting," multivalued form)

$$\psi_\theta = \prod_{i<j} \frac{(z_i - z_j)^{\theta/\pi}}{|z_i - z_j|^{\theta/\pi}} \psi_b \tag{17}$$

That is, these particles have, in some sense, a wavefunction like a Bose superfluid, corrected by their statistics. Girvin and MacDonald (*31*) showed that one can define a "hidden" ODLRO (call it ODLRO*) for the state given by Eq. 17, and further that this ODLRO* is characteristic of the QHE. This is in fact a precise way of defining the relationship between the Bose superfluid and the QHE. It is also the only known manner in which fermions—or anyons—can show a form of ODLRO in a one-body density matrix.

We observe that, if one placed the statistics of the particles of Eq. 17 into the Hamiltonian—treating them as bosons with attached flux—then their wavefunction could be directly compared with the Bose state ψ_b. If they were in the state given by Eq. 17 then the overlap would be 1, and we would call the state a QHE state. This is what we did with our numerically obtained ground states for anyons in magnetic field, finding very large overlaps with free boson states at those values of field where a QHE state is expected—that is, where the anyons are bosons at the level of mean-field theory. We also observed the expected deep minimum in the ground-state energy (Fig. 8) at the commensuration point. The fractions (anyon density/flux density) which we find are the same as those predicted by Halperin (*38*) and Haldane (*37*). This work thus confirms, by direct calculation of anyon states, the picture of the hierarchy states obtained from previous theoretical work.

Another form of anyon superfluid was recently proposed by Laughlin (*22, 28*). Since Laughlin gave the picture quite effectively in his article (*22*), we will be brief. We know that fermions can form a two-body condensate, showing a two-body form of ODLRO, in which a macroscopic number of pairs in the fluid are in a single quantum state—just as many Bose particles are in a single quantum state in a Bose superfluid. This is possible because pairs of fermions are, in some sense, bosons—for instance, if they are tightly bound in real space, then for energies small with respect to the binding energy, they are bosons. Laughlin pointed out that pairs of particles with statistics angle $\theta = \pi/2$ are also bosons—since, in the real-space picture, exchange of two bound pairs of anyons gives $(\exp i\theta)^4$—which is 1, for $\theta = \pi/2$. Generalizing this, one can imagine n-body composites for other values of n than 2, and other statistics, which may form a superfluid possessing an n-body ODLRO. Finally, if the anyons are charged, the superfluid should then be a superconductor.

Anyon superconductivity is a remarkable and ingenious idea, and it seems that it is almost certainly correct. Work in this area is presently expanding rapidly, and so we do not feel that it is possible to give an adequate review at this time. We will say that, to our knowledge, everyone (perhaps a biased set!) who has looked at the question theoretically, including ourselves, has confirmed the idea that anyon superconductivity exists in principle (*22, 23, 28, 29, 39, 40–43*), for certain rational fractional values of the statistics angle θ/π. The properties of anyon superconductors are still being worked out, although there is some agreement developing on some familiar analogs from fermion superconductors (the Meissner effect and flux quantization, for example) (*29, 40–42*).

Beyond the question of the existence in principle of anyon superconductivity is another question raised by Laughlin (*22*): does it exist in fact, in the two-dimensional, strongly magnetic planes of the high-temperature superconductors? This question, of course, remains unanswered. It awaits a detailed theory of anyon superconductors which allows for a conclusive experimental test. A number of tests have been suggested, based on signatures expected to arise from the broken time reversal symmetry (*39, 40, 44, 45*) of the anyons. Many of these signatures face the problem that they are strongly canceled out if, as is possible, the many stacked planes of the crystal choose alternating chirality. We have proposed a signature (*42*) which is independent of chirality and thus does not cancel for many stacked planes. One can say little more at this point, as the field is new and in a rapid state of flux.

We want to emphasize that the two questions mentioned above are quite separate. It is worthwhile to study the possibility and properties of anyon superconductors, whether or not they have anything to do with the current materials of interest. This point is perhaps in danger of being overlooked in the face of the current intense interest in high-temperature superconductivity.

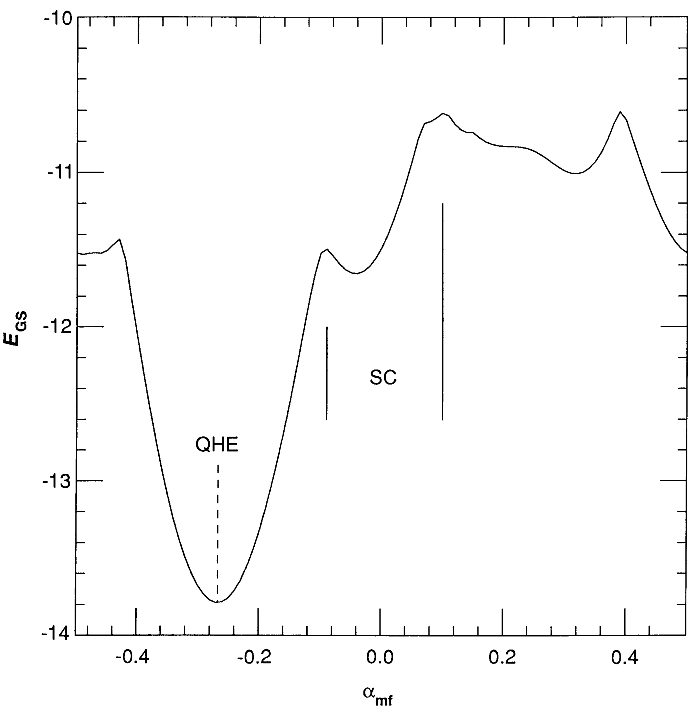


Fig. 8. Ground-state energy for eight "half-statistics" particles ($\theta = \pi/2$) as a function of magnetic field (α_{mf}). The deep minimum is the quantum Hall (QHE) state; the smaller minimum around zero field is a state which shows superconducting pairing (as evidenced by flux quantization).

Finally, we wish to offer a caveat. We have noted that the only way (so far) that we can imagine for anyons to appear in nature is as vortices in a fluid, which in turn is confined to two dimensions. We have also noted, while thinking about the hierarchy states of the QHE above, that these vortices "see" the background fluid as an effective magnetic field. Thus, we ask: is there any way to make an anyon which is not embedded in an effective magnetic field? If there is not, can the proposed superconducting state survive this effective magnetic field? We know that the QHE state can, since it prefers a certain magnetic field and tolerates some deviation from the preferred value. We raise these questions because they seem to be fundamentally important, and they have not been addressed in theories of anyon superconductivity. A small part of an answer might be obtained from Fig. 8. The horizontal axis is magnetic field, and the small dip in the center is the "superconducting" state (we use quotation marks since there are only eight particles). The large dip at the left is the QHE state, which, in all our simulations to date, is considerably more robust than the superconducting state in the face of variations in magnetic field.

Conclusions

It is clear that the quasiparticles of the FQHE can be viewed as anyons—at sufficiently large separations that they do not significantly interpenetrate, and at sufficiently low energies that their parent fluid remains two dimensional. Further work is needed to clarify the limits within which the anyon model gives the correct physics. This is similar to the physics of helium atoms, which may be modeled as repulsive bosons as long as they do not interpenetrate too much, or ionize. The crucial and very interesting difference is, of course, that the repulsive Bose gas is fairly well understood—while the study of the anyon gas is still largely ahead of us. We note that, while an unambiguous experimental demonstration of the fractional statistics of the quasiparticles is probably extremely difficult, there is already some interesting evidence for their fractional charge (*46*).

We have noted a theoretical obstacle to the realization of anyon superconductivity: that anyons which are vortices (as are all candidates to date) feel a large (fictitious) magnetic field due to the background fluid in which they move—a field which is, in general, too large to allow a superconducting ground state. This problem may perhaps be resolved by, for example, some other mechanism for generating fractional statistics, or by a lattice commensuration which renders the fictitious field innocuous. Assuming that this obstacle can be overcome in some instances, there remains the practical question of the realization of anyon superconductivity in the laboratory. There is currently a vigorous theoretical effort towards a better understanding of the properties of anyon superconductors—in particular, those properties which might serve as an unambiguous signal of the broken time-reversal symmetry of the anyon state. Such an understanding is needed as a guide to the experimental search for this new type of superfluid. Perhaps, as suggested by Laughlin (*22*), we will find that anyon superconductivity has already been realized in the high-temperature superconductors; certainly the question deserves serious attention. In the event of a negative answer, the possibility of anyon superconductivity, once raised, will stand as a challenge to physicists from now on. We need only recall the First Law of Physics: "whatever is not forbidden is compulsory."

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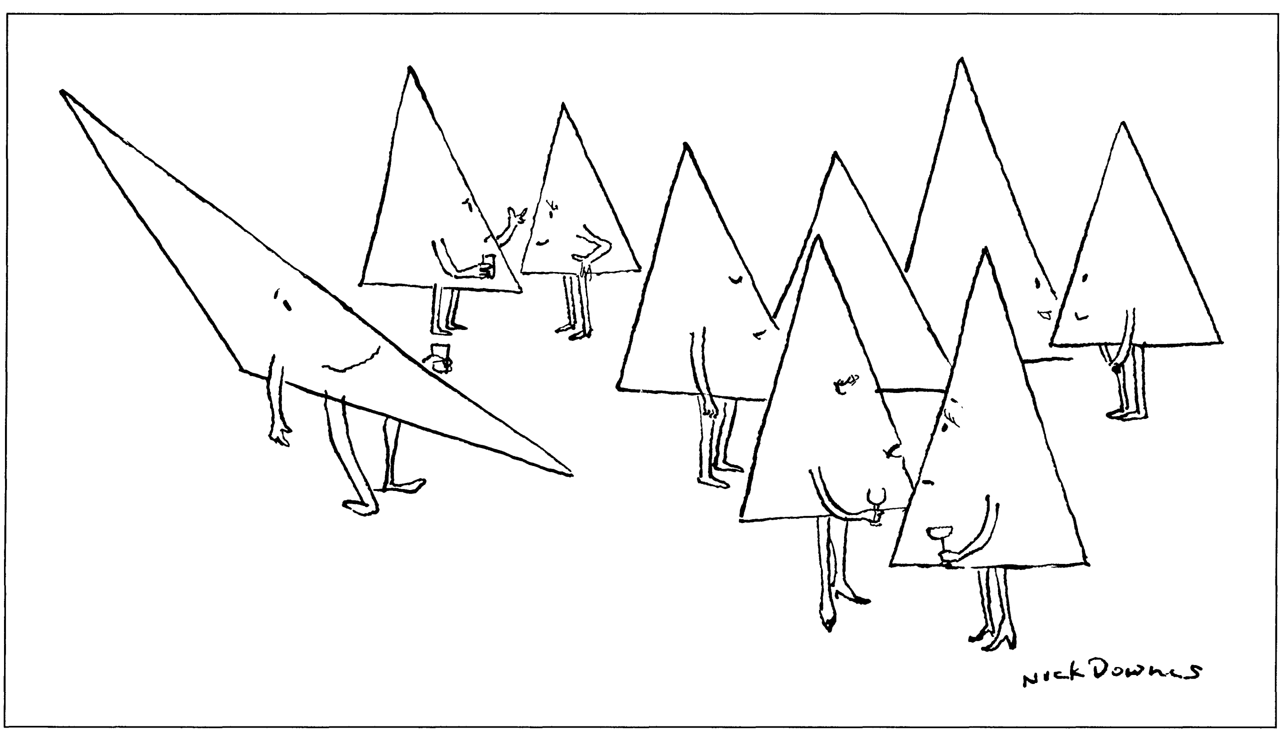

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