To Have and Have Knot: When Are Knots Alike?

Knots might seem like a silly toy for grown men, but these mathematicians aren't just stringing people along. Their result ties up some 80-year-old loose ends

TWO MATHEMATICIANS HAVE SETTLED an 80-year-old conjecture in the theory of knots: Two knots are the same if and only if the space around them is the same. The theorem, proved by Cameron Gordon of the University of Texas at Austin and John Luecke of the Courant Institute in New York, shows that no essential information is lost by throwing away the knot and distorting the space around it. "It's arguably the best theorem yet proven in the theory of knots," according to Martin Scharlemann, a mathematician at the University of California at Santa Barbara.

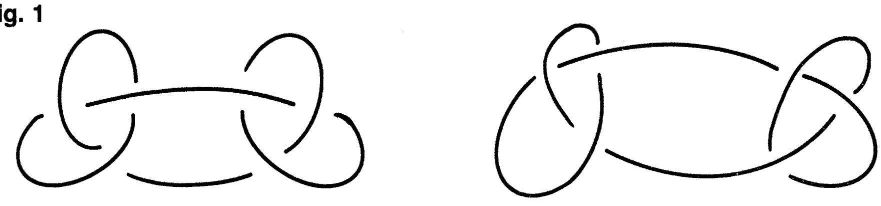
A mathematical knot is a curve that meanders smoothly through ordinary three-dimensional space and that satisfies two properties: One, it is "closed," that is, it circles around on itself with no beginning or end; and two, it is "simple," meaning that it does not intersect itself. A mathematical knot is easy to visualize because it is just a formal, mathematical version of a knot in a piece of string. Take a length of string, knot it up in any way you like, then fuse the two ends together, and you have a representation of a mathematical knot. (Mathematicians insist on fusing the ends together because this freezes the knot in place; as long as the ends are free, one can untie the knot, but once the ends are fused, the only way to untie it is to cut it first.)

A major question mathematicians ask about knots is whether two knots that look different are actually the same knot, just stretched and twisted in different ways. For instance, a rubber band fresh out of the box looks like the trivial circle knot (which is also known as the "unknot" because it is the result of taking a length of string and tying no knots in it, then fusing the ends of the string together). A rubber band that is a tangled mess from being stretched and twisted and jumbled does not look the same as the circle knot, but it really is—if you are careful and skillful, you can work the kinks out of the tangled mess until once again it looks like the circle knot.

Mathematicians say two knots are "homeomorphic" if they are the same knot in the above sense—once they are untangled, they look the same. The central problem in knot theory is to distinguish one type of knot from another and organize them into classes of homeomorphic knots. However, one tangled mess looks much like another—how can you tell what kind of knot you've got?

One of the key techniques is to focus not on the knot itself, but on the space surrounding it, the so-called complement to the knots. One can picture the complement as a plaster mold of the knot: Start with a knotted string and pour plaster around it, then remove the string after the plaster is hardened. What is left is a solid piece of plaster with a tiny looping tunnel running through it. This plaster mold represents the complement to the original knot.

Fig. 1



Square knot   Granny knot

By studying the complement to the knot instead of the knot itself, mathematicians can bring to bear a number of powerful tools from the field of topology, a branch of mathematics concerned with abstract properties of shape and continuity. (A topologist is said to be a mathematician who cannot tell the difference between a doughnut and a coffee cup; each is a solid with a hole you can stick your finger through.) The complement, which is three dimensional, carries a richer topological structure than the knot itself, which is one dimensional. The topological structure of the complement necessarily contains certain information about the knot; what Gordon and Luecke have shown is that it contains all the essential information about the knot.

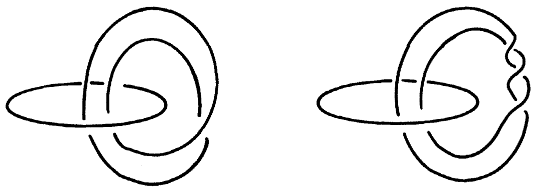
If two knots are homeomorphic, then their complements must also be homeomorphic because the deformation that transforms one knot into the other is required to be a deformation on all of the surrounding space, not just the knot itself. (If the deformations were defined only on the knot itself, then every knot would be homeomorphic to a simple circular loop by tightening the tangled part of the knot down to a single point.) Another way to say the same thing is that if two complements are inequivalent, then so are the corresponding knots. Thus a mathematician can prove that two knots are not the same simply by proving that their complements are not the same. This makes it possible, for instance, to prove that the square knot and the granny knot (see Fig. 1) are distinct—a fact unsurprising to scouts or sailors, but less obvious to mathematicians.

The converse question remains: If two complements are the same, does this imply that the knots are the same? Gordon and Luecke have proved the answer is yes. The question goes back to 1908 and the Austro-German topologist Heinrich Tietze, but progress has come only recently. In 1985, Gordon and Luecke in collaboration with Marc Culler and Peter Shalen, both now at the University of Illinois at Chicago, proved that no more than two distinct knots could have the same complement. The new result ties up the last loose ends of the problem.

At first glance, the theorem seems trivially true—if you can deform one complement into the other, it seems obvious that the knot is carried along for free."It sounds so plausible, it's hard to believe it's deep," says Ralph Krause, the National Science Foundation project director for topology and foundations. The most convincing argument against the theorem's triviality is the fact that the analogous statement for links, which consist of two or more knots linked together, is false. For example, the two links shown in Fig. 2 have homeomorphic complements, but the links themselves are not homeomorphic. (The reader for whom this is obvious, however, should consider a career in topology.)

Gordon and Luecke's proof is far from trivial, occupying the better part of a 59-page manuscript. As is common in mathematics, they actually prove a more difficult result which has the main theorem as an "easy" consequence. The deeper result has a

Fig. 2



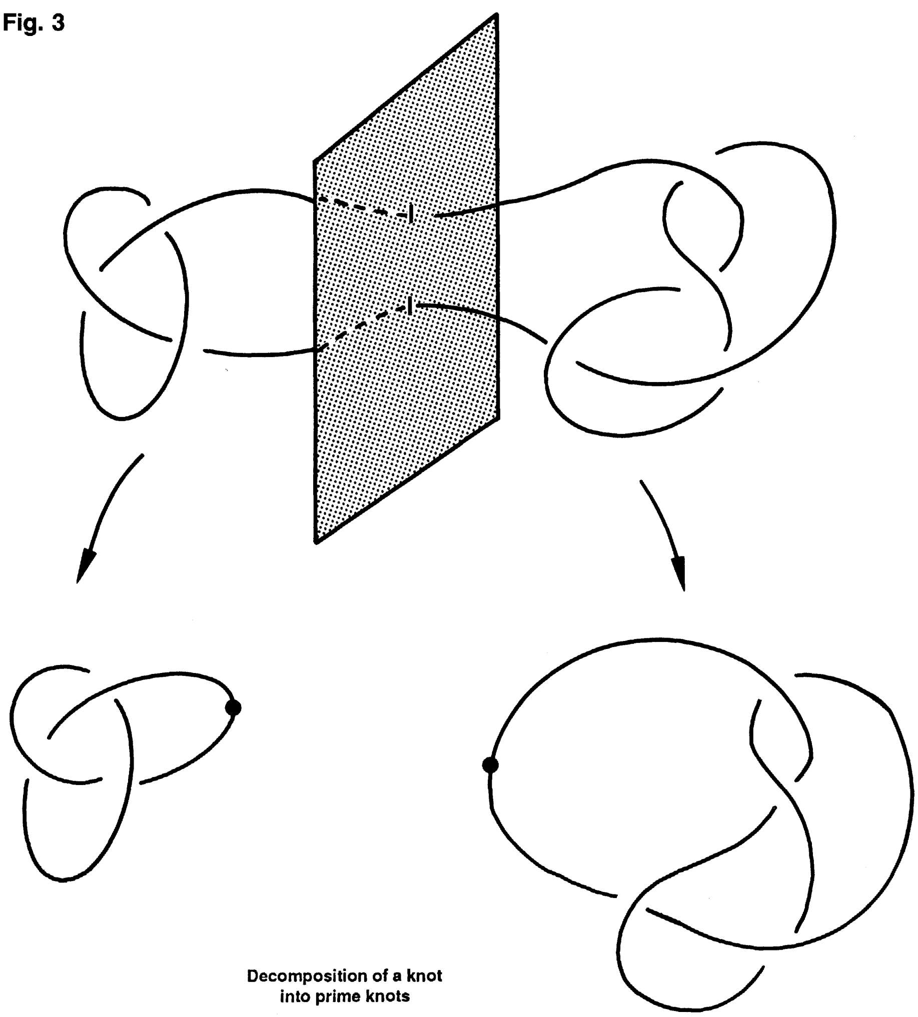
Two nonhomeomorphic links with homeomorphic complements

nice negative ring to it: Nontrivial Dehn surgery on a nontrivial knot never yields the three-sphere. Each of the terms in this complicated-seeming statement is actually easy to understand. Dehn surgery is a process by which a knot is fattened up into a tube, removed from the complement, and then stitched back in. The surgery is trivial if the tube is stitched back in as it was taken out; nontrivial surgery requires some kind of twisting of the tube. (No anesthetic is used in either case.) A nontrivial knot is one that is not homeomorphic to the circle. The three-sphere is a topological space equivalent to ordinary three-dimensional space with an extra "point at infinity" tacked on; it is the three-dimensional analogue to the more familiar two-dimensional sphere.

Just as the circle and the two-dimensional sphere are very simple objects, so is the three-sphere (to a topologist, anyway). Gordon and Luecke's theorem says, in essence, that if you start with a complicated object (the nontrivial knot) and perform a complicated operation on it (nontrivial Dehn surgery), then you will never get the simple three-sphere.

Thanks to related work of Wilbur Whitten at the University of Southwestern Louisiana, the theorem has an extra payoff for a class of knots known as prime knots. Prime knots are analogous to prime numbers. If a knot is positioned so that it pierces a plane in exactly two points, then the knot can be mathematically decomposed into two simpler knots by pinching the two points together (see Fig. 3). If one of the simpler knots is always the unknot, then the original knot is said to be prime, just as an integer is called prime if it can only be factored into itself and 1. The analogy goes a bit further: Just as every integer can be decomposed as a product of primes, it turns out that every knot can be decomposed into prime knots, with the same collection of prime knots being obtained no matter how the decomposition is performed.

Fig. 3



Decomposition of a knot into prime knots

Gordon and Luecke's theorem, in conjunction with a theorem of Whitten's, implies that two prime knots are homeomorphic if and only if a certain algebraic structure associated to each knot is the same. That algebraic structure, known as the fundamental group, is actually associated to the complement of the knot, and every knot has one, not just prime knots. The fundamental group expresses some of the topology of the complement in algebraic language, which makes it possible to compare different knots by comparing their algebraic descriptions. In general, the complement of a knot that has a complicated topology will have a complicated fundamental group. Ordinary three-space, the simplest three-dimensional space, has a group with only one element, while the complement of the unknot (three-space with a circle removed from it) has the integers as its group.

Homeomorphic complements necessarily have identical ("isomorphic," in the argot) groups, but the converse in this case is not always true—the granny knot and the square knot, for instance, have distinct complements but isomorphic groups. However, Whitten showed in 1985 that when one considers only prime knots then the converse is true: Prime knots with isomorphic groups have homeomorphic complements. When Whitten's result is added to the theorem of Gordon and Luecke, one gets a simple way to tell whether or not two prime knots are distinct: If two prime knots have fundamental groups that are the same (isomorphic), then the prime knots themselves are the same (homeomorphic).

Gordon and Luecke's solution to Tietze's 80-year-old problem has no immediate practical or even theoretical applications. The main motivation, Gordon says, was to answer a basic question about knots that was hard enough to last for eight decades. However, the techniques developed to prove their theorem may succeed in settling other difficult problems in Dehn surgery. It would also be interesting, Gordon adds, to see what might be true for links, where the theorem itself is patently false.

■ BARRY A. CIPRA