Computer-Drawn Pictures Stalk the Wild Trajectory

Even simple systems can exhibit chaotic behavior, but tracking them mathematically can be tricky; computer-generated pictures can help in the pursuit

A COMPUTER-GENERATED PICTURE is worth as many as 10 million words, according to three mathematicians studying the young science of chaos. James Yorke and Celso Grebogi, both at the University of Maryland, and Stephen Hammel, at the Naval Surface Weapons Center in Silver Spring, Maryland, have devised a method by which a computer can check how far its calculation of a chaotic trajectory can be trusted. They have applied their method to produce "picture proofs" for two of the benchmark objects in chaos theory, known as the Ikeda map and the Hénon map (see box).

Chaos theory is a relatively new field which studies the complicated dynamics that may reside in even very simple mathematical models. Researchers believe that chaos theory offers a mathematical framework for understanding much of the noise and turbulence that is seen in experimental science. Phenomena as diverse as weather patterns, population dynamics, chemical reactions, and laser optics sport chaotic features that the theory may shed light on. Even the simple pendulum can be "kicked" into displaying chaotic behavior.

The foundations of chaos theory were laid nearly a century ago, mainly by the French mathematician Henri Poincaré. However, only recently have investigators been able to take a "hands on" approach to the subject, mainly through the development of high-speed computers. And these days it is possible to study chaos on anything from a supercomputer down to a simple pocket calculator; all it takes is the ability—and patience—to do one mindless calculation after another.

A "trajectory" in chaos theory is a sequence of points in the plane, in which each point produces its successor according to some mathematical function. The Hénon map, for instance, defines the point (x_{n+1}, y_{n+1}) by the equations $x_{n+1} = 1.4 + 0.3y_n - x_n^2, y_{n+1} = x_n$. Depending on the function and the starting point, a trajectory may escape off to infinity; or it may be drawn toward a single point or toward some regular periodic motion; or it may behave chaotically, jumping about in seemingly random fashion.

A chaotic trajectory exhibits three features. First, the motion stays within a bounded region—it does not get larger and larger without limit. Second, the trajectory never settles into any kind of periodic pattern, though it will repeatedly "visit" a certain set of points. (A point is "visited" if every circle centered at the point, no matter how small, contains infinitely many points of the trajectory.) The set of points visited

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by a trajectory is known as the trajectory's attractor. One of the strange—and enticing—aspects of the subject is that in many cases, such as the Hénon map, every trajectory appears to have the same attractor, in which case one may speak of the *system* as having an attractor. Mathematicians are spending a lot of time proving that systems that look chaotic truly are chaotic.

The third feature characterizing chaotic trajectories is a "sensitivity to initial conditions," which, loosely speaking, means that trajectories starting close together separate at an exponential rate. The upshot of this is that any kind of error—in particular, roundoff error—propagates in the course of computation so that the developing trajectory soon bears no relationship whatsoever to the true trajectory of the starting point. An error as small as one part in a trillion will grow to dominate the computation in as few as a couple dozen steps.

Because of this feature, computer-generated chaotic trajectories have been viewed with some suspicion: do these graphs correspond to real trajectories or are they mere machine artifacts? It is this question that Yorke, Grebogi, and Hammel address in an article to appear in the *Bulletin of the American Mathematical Society*. They have shown how the computer itself can verify that its computed trajectory does not stray from a true trajectory by more than one part in 10 million for as many as 10 million steps in the process.

Their result, at first glance, is paradoxical. The computed trajectory of, say, the Hénon map starting at $x_0 = y_0 = 0$, most certainly is *not* the true trajectory or anywhere close to it. Not even a high-precision supercomputer can stay close to the true trajectory; in one experiment, a Cray 1 and a Cyber 205 gave wildly different answers by just the 50th point in the trajectory, due to the different roundoff mechanisms of the two machines. So how is it possible to claim (much less prove!) that 10 *million* points stay close to a true trajectory?

The explanation lies in the difference between the definite article "the" and the indefinite "a." The computed trajectory does not stay close to *the* true trajectory of *the*

Fig. 1.

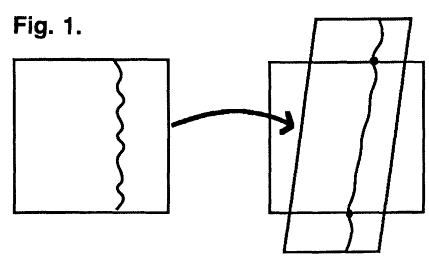
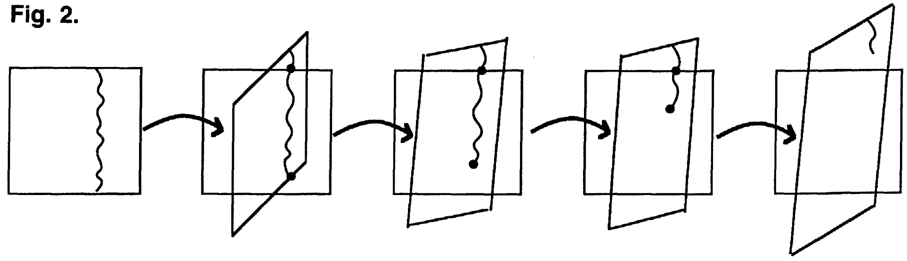


Fig. 2.



initial point; rather it stays close to *a* true trajectory that starts at *some* nearby initial point. The only catch is, you do not know which point; you only know that it is within one part in 10 million of where you thought you started. (The exact estimates depend on the computer's numerical precision. Roughly speaking, if the computer calculates with $2n$ digits accuracy, then the numerical trajectory can be expected to approximate a true trajectory with n digits accuracy for as many as $10n$ points.)

The verification method is almost brutally simple. As it calculates a trajectory, the computer places a numerical box about each point, with sides approximately one-millionth long. (The trick is to orient each box at a suitable angle.) When the chaotic process maps a point on the trajectory to the next point, it also maps the box about the first point into a region that is very close to a parallelogram in shape. In general, the parallelogram has expanded in one direction and contracted in the other. The key point is to verify that, even when the minuscule errors are allowed for, the parallelogram coming from the first box and the new box about the second point form a shape like a fat plus sign (see Fig. 1). This guarantees that any curve connecting the top and bottom of the first box will "survive" as a curve connecting the top and bottom of the second box. (The computer does not look at these "theoretical" curves; its job was done when it checked for the plus sign shapes.)

If the plus sign shape is verified for the first 10 million steps, then the result can be read backward: any point on the final surviving curve came from some point on the curve before it, which came from the point before it, and so on back to the very first box. This proves in fact that infinitely many true trajectories stay "boxed in" near to the numerical trajectory for at least the first 10 million steps.

The verification stops if the parallelogram fails to make a plus sign with the next box. This rarely happens, but it only has to happen once (see Fig. 2). Yorke says such "glitches" almost certainly occur, but only in a simpler one-dimensional setting has it been possible to prove that they do.

Part of the appeal of chaos lies in the pretty pictures that can arise from simple mathematical formulas. Now, Yorke says, some of those pictures can aspire to the status of proof. It is common in mathematical writing to omit the details of a calculation. In this case, Yorke notes, the "proofs" omit some 50 million lines of detail!

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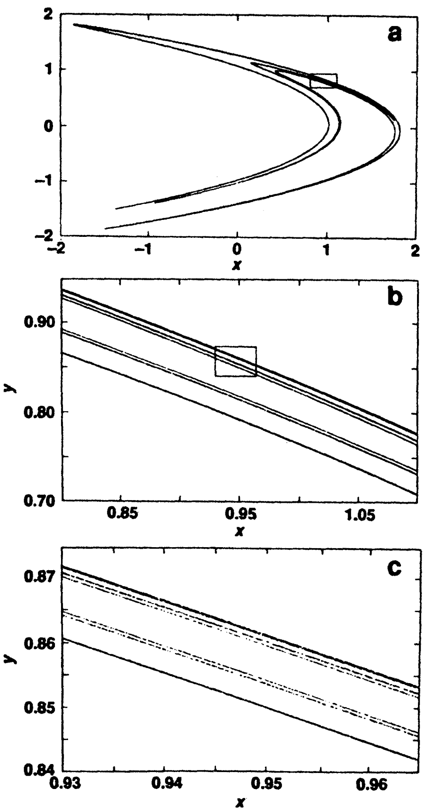
Fatal Attractions?

The Hénon attractor grows on the computer screen from a single point into a graceful boomerang shape. The image is produced by iterating a pair of simple functions, one of them having a nonlinear term, applied initially to some point in the plane specified by particular x- and y-coordinates. With each iteration, the attractor as a whole is stretched and folded, in much the same way that a baker kneads bread dough. Points that are initially close together get stretched apart, but are eventually folded back close together. All that's missing is the yeast.



There is much that is strange about the Hénon attractor. For one thing, it appears to be the same regardless of the iteration's starting point. It also seems to retain complex detail at all levels: magnify any "line" and it is seen to separate into several parallel lines, each of which also separates under further magnification.

But is this strangeness for real? At this point no one is certain. The accumulated computational evidence favors strangeness, and the result of Yorke *et al.* certifies that the computer results are accurate for upwards of 10 million points. But the theoretical issue lies not in the first 10 million points of a trajectory, or in the first 10 billion points, or in the first ten anything; the theoretical issue lies in the infinite limits of the trajectories—and precious little is known about those limits.

It *is* known that individual trajectories have attractors, simply because they are confined to a bounded part of the plane. But different trajectories may have different attractors, and their behavior at the attractors may be periodic rather than chaotic. The "true" trajectories that are being approximated by computation may simply not have settled into a periodic orbit, or their periodic orbit may simply include an astronomical (or an Avogadro's) number of points.

The prevailing opinion among theorists is that the Hénon map has a single attractor that is both chaotic and strange. (More precisely, "almost all" trajectories are believed to have the same attractor. "Almost all" is an analytic incantation that has a precise mathematical meaning. One sign of chaos theory's youth, though, is that many other terms, including "chaotic" and "strange," have not settled into generally agreed upon definitions.) Others emphasize the doubts. Gregory Davis of the University of Wisconsin at Green Bay is exploring the possibility that the Hénon map has infinitely many coexisting periodic attractors. His preliminary results are consistent with the computational data: for two points to be drawn into the same periodic orbit, they would have to start closer together than current computations are able to resolve. Davis speculates that "what is seen on the computer screen is a point that is bouncing in and out of different basins of attraction, thereby giving the effect of a 'strange attractor'."

The theoretical issues that are unresolved for the Hénon map are also up for grabs for many other ostensibly strange systems. The Hénon map has attracted attention partly because of its comparative simplicity—its complex behavior arises out of a simple quadratic term in the equations defining the map, whereas other maps rely on more complicated nonlinear expressions. "Simplicity," however, has not made theoretical life much easier. One of the things he has learned about the subject, Yorke says, is "the more you understand it, the harder it seems." ■ **B. A. C.**