Numerical Evidence That the Motion of Pluto Is Chaotic

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The Digital Orrery has been used to perform an integration of the motion of the outer planets for 845 million years. This integration indicates that the long-term motion of the planet Pluto is chaotic. Nearby trajectories diverge exponentially with an *e*-folding time of only about 20 million years.

THE DETERMINATION OF THE STABILITY OF THE SOLAR system is one of the oldest problems in dynamical astronomy, but despite considerable attention all attempts to prove the stability of the system have failed. Arnold has shown that a large proportion of possible solar systems are quasiperiodic if the masses, and orbital eccentricities and inclinations, of the planets are sufficiently small (*1*). The actual solar system, however, does not meet the stringent requirements of the proof. Certainly, the great age of the solar system suggests a high level of stability, but the nature of the long-term motion remains undetermined. The apparent analytical complexity of the problem has led us to investigate the stability by means of numerical models. We have investigated the long-term stability of the solar system through an 845-million-year numerical integration of the five outermost planets with the Digital Orrery (*2*), a special-purpose computer for studying planetary motion.

Pluto's orbit is unique among the planets. It is both highly eccentric ($e \approx 0.25$) and highly inclined ($i \approx 16°$). The orbits of Pluto and Neptune cross one another, a condition permitted only by the libration of a resonant argument associated with the 3:2 commensurability between the orbital periods of Pluto and Neptune. This resonance, which has a libration period near 20,000 years (*3*), ensures that Pluto is far from perihelion when Pluto and Neptune are in conjunction. Pluto also participates in a resonance involving its argument of perihelion, the angle between the ascending node and the perihelion, which librates about $\pi/2$ with a period of 3.8 million years (*4*). This resonance guarantees that the perihelion of Pluto's orbit is far from the line of intersection of the orbital planes of Pluto and Neptune, further ensuring that close encounters are avoided.

We found in our 200-million-year integrations of the outer planets (*5*) that Pluto's orbit also undergoes significant variations on much longer time scales. The libration of the argument of perihelion is modulated with a period of 34 million years, and $h = e \sin \tilde{\omega}$, where e is the eccentricity and $\tilde{\omega}$ is the longitude of perihelion, shows significant long-period variations with a period of 137 million years. The appearance of the new 34-million-year period might have been expected, because Pluto must have two independent long-period frequencies, but the 137-million-year period was completely unexpected. It results from a near commensurability between the frequency of circulation of Pluto's ascending node and one of the principal secular frequencies of the massive planets. Pluto also participates in two other resonances involving the frequency of oscillation of the argument of perihelion and the principal secular frequencies. In our 200-million-year integration Pluto's inclination appeared to have even longer periods or possibly a secular decrease.

The similarity of Pluto's peculiar highly eccentric and inclined orbit to chaotic asteroid orbits (*6*), together with the very long periods, Pluto's participation in a large number of resonances, and the possible secular decline in inclination compelled us to carry out longer integrations of the outer planets to clarify the nature of the long-term evolution of Pluto. Our new numerical integration indicates that in fact the motion of the planet Pluto is chaotic.

Deterministic chaotic behavior. In most conservative dynamical systems Newton's equations have both regular solutions and chaotic solutions. For some initial conditions the motion is quasiperiodic; for others the motion is chaotic. Chaotic behavior is distinguished

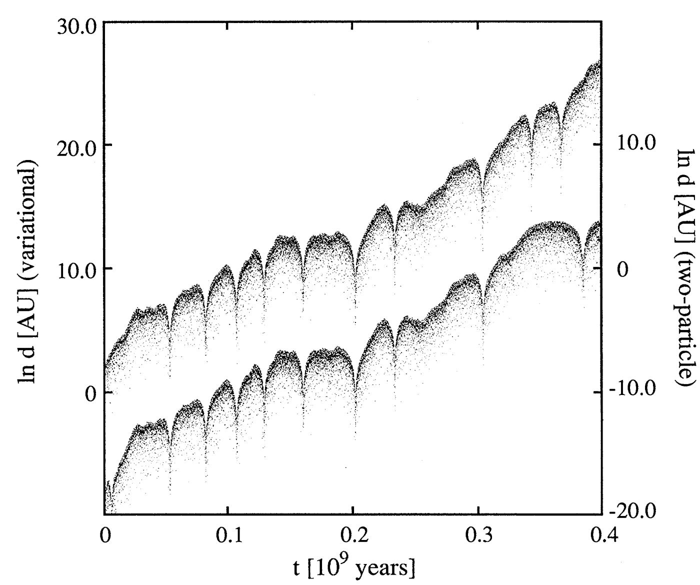

Fig. 1. The exponential divergence of nearby trajectories is indicated by the average linear growth of the logarithms of the distance measures as a function of time. In the upper trace we see the growth of the variational distance around a reference trajectory (left vertical axis). In the lower trace we see how two Plutos diverge with time (right vertical axis). The distance saturates near 45 AU; note that the semimajor axis of Pluto's orbit is about 40 AU. The variational method of studying neighboring trajectories does not have the problem of saturation. Note that the two methods are in excellent agreement until the two-trajectory method has nearly saturated.

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from quasiperiodic behavior by the way in which nearby trajectories diverge (*6, 7*). Nearby quasiperiodic trajectories diverge linearly with time, on average, whereas nearby chaotic trajectories diverge exponentially with time. Quasiperiodic motion can be reduced to motion on a multidimensional torus; the frequency spectrum of quasiperiodic motion has as many independent frequencies as degrees of freedom. The frequency spectrum of chaotic motion is more complicated, usually appearing to have a broad-band component.

The Lyapunov exponents measure the average rates of exponential divergence of nearby orbits. The Lyapunov exponents are limits for large time of the quantity $\gamma = \ln(d/d_0)/(t - t_0)$, where d is the distance in phase space between the trajectory and an infinitesimally nearby test trajectory, and t is the time. For any particular trajectory of an n-dimensional system there can be n distinct Lyapunov exponents, depending on the phase-space direction from the reference trajectory to the test trajectory. In Hamiltonian systems the Lyapunov exponents are paired; for each non-negative exponent there is a non-positive exponent with equal magnitude. Thus an m-degree-of-freedom Hamiltonian system can have at most m positive exponents. For chaotic trajectories the largest Lyapunov exponent is positive; for quasiperiodic trajectories all of the Lyapunov exponents are zero.

Lyapunov exponents can be estimated from the time evolution of the phase-space distance between a reference trajectory and nearby test trajectories (*7, 8*). The most straightforward approach is to simply follow the trajectories of a small cloud of particles started with nearly the same initial conditions. With a sufficiently long integration we can determine if the distances between the particles in the cloud diverge exponentially or linearly. If the divergence is exponential, then for each pair of particles in the cloud we obtain an estimate of the largest Lyapunov exponent. With this method the trajectories eventually diverge so much that they no longer sample the same neighborhood of the phase space. We could fix this by periodically rescaling the cloud to be near the reference trajectory, but we can even more directly study the behavior of trajectories in the neighborhood of a reference trajectory by integrating the variational equations along with the reference trajectory. In particular, let $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ be an autonomous system of first-order ordinary differential equations and $\mathbf{y}(t)$ be the reference trajectory. We define a phase-space variational trajectory $\mathbf{y} + \delta\mathbf{y}$ and note that $\delta\mathbf{y}$ satisfies a linear system of first-order ordinary differential equations with coefficients that depend on $\mathbf{y}(t)$, $\delta\mathbf{y}' = \mathbf{J}\cdot\delta\mathbf{y}$, where the elements of the Jacobian matrix are $J_{ij} = \partial f_i/\partial y_j$.

Our numerical experiment. For many years the longest direct integration of the outer planets was the 1-million-year integration of Cohen, Hubbard, and Oesterwinter (*9*). Recently several longer integrations of the outer planets have been performed (*5, 10, 11*). The longest was our set of 200-million-year integrations. Our new 845-million-year integration is significantly longer and more accurate than all previously reported long-term integrations.

In our new integration of the motion of the outer planets the masses and initial conditions are the same as those used in our 200-million-year integrations of the outer planets. The reference frame is the invariable frame of Cohen, Hubbard, and Oesterwinter. The planet Pluto is taken to be a zero-mass test particle. We continue to neglect the effects of the inner four planets, the mass lost by the Sun as a result of electromagnetic radiation and solar wind, and general relativity. The most serious limitation of our integration is our ignorance of the true masses and initial conditions. Nevertheless, we believe that our model is sufficiently representative of the actual solar system that its study sheds light on the question of stability of the solar system. To draw more rigorous conclusions, we must determine the sensitivity of our conclusions to the uncertainties in masses and initial conditions, and to unmodeled effects.

Our earlier integrations were limited to 100 million years forward and backward in time because of the accumulation of error, which was most seriously manifested in an accumulated longitude error of Jupiter of order 50°. In our new integrations we continue to use the 12th-order Störmer predictor (*12*), but a judicious choice of step size has reduced the numerical errors by several orders of magnitude. In all of our integrations the error in energy of the system varies nearly linearly with time. In the regime where neither roundoff nor

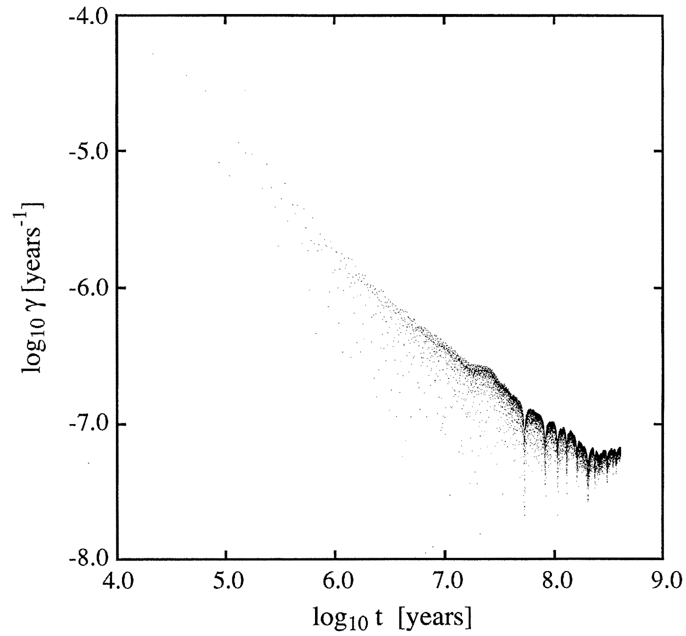

Fig. 2. The conventional representation of the Lyapunov exponent calculation, the logarithm of γ versus the logarithm of time. Convergence to a positive exponent is indicated by a leveling off; for regular trajectories this plot approaches a line with slope minus one.

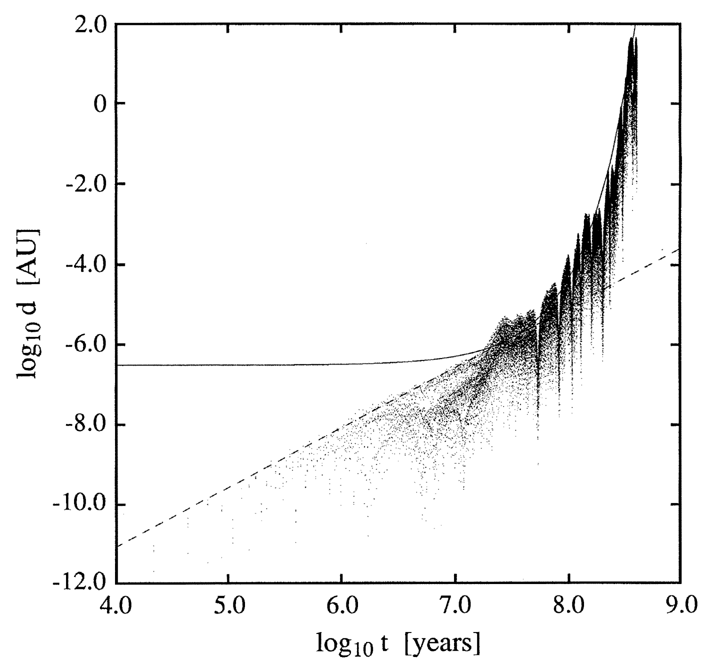

Fig. 3. Common logarithm of the distance between several pairs of Plutos, in AU, versus the common logarithm of the time, in years. The initial segment of the graph closely fits a 3/2 power law (dashed line). The solid line is an exponential chosen to fit the long-time divergence of Plutos. The exponential growth takes over when its slope exceeds the slope of the power law.

truncation error is dominant the slope of the energy error as a function of time depends on step size in a complicated way. For some step sizes the energy error has a positive slope; for others the slope is negative. This suggests that there might be special step sizes for which there is no linear growth of energy error. By a series of numerical experiments we indeed found that there are values of the step size where the slope of the linear trend of energy vanishes. The special step sizes become better defined as the integration interval of the experiments is increased.

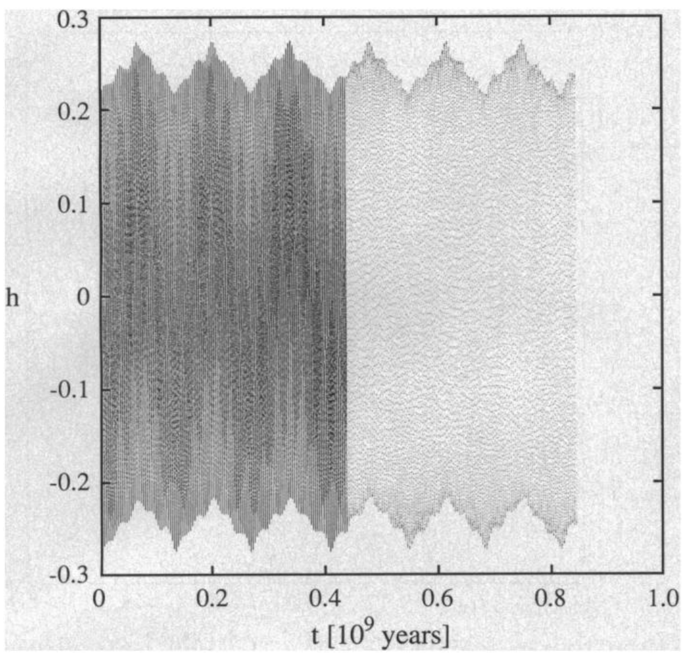

Fig. 4. The orbital element $h = e \sin \tilde{\omega}$ for Pluto over 845 million years. On this scale the dominant period (the 3.7-million-year circulation of the longitude of perihelion) is barely resolved. The most obvious component has a period of 137 million years. The sampling interval was increased in the second half of our integration.

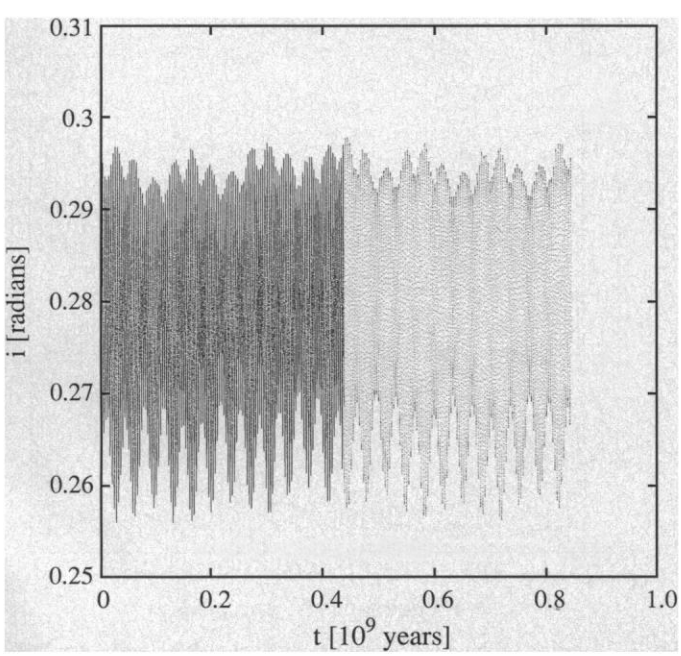

Fig. 5. The inclination i of Pluto over 845 million years. Besides the 34-million-year component and the 150-million-year component there appears to be a component with a period near 600 million years.

We chose our step size on the basis of a dozen 3-million-year integrations, and numerous shorter integrations. For our new long integration we chose the step size to be 32.7 days. This seemingly innocuous change from a step size near 40 days dramatically reduces the slope of the energy error, by roughly three orders of magnitude. If the numerical integration were truncation error–dominated, for which the accumulated error is proportional to h^n, where h is the step size and n is the order of the integrator, then this reduction of step size would improve the accumulated error by only about a factor of 10.

In our new integration the relative energy error (energy minus initial energy divided by the magnitude of the initial energy) accumulated over 845 million years is -2.6×10^{-10}; the growth of the relative energy error is still very nearly linear with a slope of -3.0×10^{-19} year^{-1}. By comparison the rate of growth of the relative energy error in our 200-million-year integrations was 1.8×10^{-16} year^{-1}. The errors in other integrations of the outer solar system were comparable to the errors in our 200-million-year integrations. The rate of growth of energy error in the 1-million-year integration of Cohen, Hubbard, and Oesterwinter was 2.4×10^{-16} year^{-1}. For the 6-million-year integration of Kinoshita and Nakai (*10*) the relative energy error was approximately 5×10^{-16} year^{-1}. For the LONGSTOP integration the growth of relative energy (as defined in this article) was -2.5×10^{-16} year^{-1}. Thus the rate of growth of energy error in the integration reported here is smaller than all previous long-term integrations of the outer planets by a factor of about 600.

We verified that this improvement in energy conservation was reflected in a corresponding improvement in position and velocity errors by integrating the outer planets forward 3 million years and then backward to recover the initial conditions, over a range of step sizes. For the chosen step size of 32.7 days the error in recovering the initial positions of each of the planets is of order 10^{-5} astronomical units (AU) or about 1500 km. Note that Jupiter has in this time traveled 2.5×10^{15} km.

The error in the longtitude of Jupiter can be estimated if we assume that the energy error is mainly in the orbit of Jupiter. The relative energy error is proportional to the relative error in orbital frequency so the error in longitude is proportional to the integral of the relative energy error: $\Delta\lambda \approx tn\Delta E(t)/E$, where n is the mean motion of Jupiter and t is the time of integration. Because the energy error grows linearly with time the position error grows with the square of the time. The accumulated error in the longitude of Jupiter after 100 million years is only about 4 arc minutes. This is to be compared with the 50° accumulated error estimated for our 200-million-year integrations. The error in the longitude of Jupiter after the full 845 million years is about 5°.

We have directly measured the integration error in the determination of the position of Pluto by integrating forward and backward over intervals as long as 3 million years to determine how well we can reproduce the initial conditions. Over such short intervals the round-trip error in the position of Pluto grows as a power of the time with an exponent near 2. The error in position is approximately $1.3 \times 10^{-19} t^2$ AU (where t is in years). This growth of error is almost entirely in the integration of Pluto's orbit; the round-trip error is roughly the same when we integrate the whole system and when we integrate Pluto in the field of the Sun only. It is interesting to note that in the integrations with the 32.7-day step size the position errors in all the planets are comparable. Extrapolation of the round-trip error for Pluto over the full 845-million-year integration gives an error in longitude of less than 10 arc minutes.

Lyapunov exponent of Pluto. We estimated the largest Lyapunov exponent of Pluto by both the variational and the phase-space distance methods during the second half of our 845-million-year run. Figure 1 shows the logarithm of the divergence of the phase-space distance in a representative two-particle experiment and the growth of the logarithm of the variational phase-space distance. We measured the phase-space distance by the ordinary Euclidean norm in the six-dimensional space with position and velocity coordinates. We measured position in AU and velocity in AU/day. Because the magnitude of the velocity in these units is small compared to the magnitude of the position, the phase-space distance is effectively equivalent to the positional distance, and we refer to phase-space distances in terms of AU. For both traces in this plot the average growth is linear, indicating exponential divergence of nearby trajectories with an *e*-folding time of approximately 20 million years. The shapes of these graphs are remarkably similar until the two-particle divergence grows to about 1 AU, verifying that the motion in the neighborhood of Pluto is properly represented. A more conservative representation of this data is to plot the logarithm of γ versus the logarithm of time (Fig. 2). The leveling off of this graph indicates a positive Lyapunov exponent.

To study the details of the divergence of nearby trajectories we expand the early portion of the two-particle divergence graph (Fig. 3). The separation between particles starts out as a power law with an exponent near 3/2. The square law we described earlier estimates the actual total error, including systematic errors in the integration process. The 3/2 power law describes the divergence of trajectories subject to the same systematic errors. Only after some time does the exponential take off. The power law is dominated by the exponential only after the rate of growth of the exponential exceeds the rate of growth of the power law. This suggests that the portion of the divergence of nearby trajectories that results only from the numerical error fits a 3/2 power law and that this error "seeds" the exponential divergence that is the hallmark of chaos. We tested this hypothesis by integrating a cloud of test particles with the orbital elements of Pluto in the field of the Sun alone. The divergence of these Kepler "Plutos" grows as $3.16 \times 10^{-17} t^{3/2}$ AU. This is identical to the initial divergence of the Plutos in the complete dynamical system, showing that two-body numerical error completely accounts for the initial divergence.

Only the second half of the integration was used in the computation of the Lyapunov exponents, because the measurement in the first half of our integration was contaminated by over-vigorous application of the rescaling method, and gave a Lyapunov exponent about a factor of 4 too large. The rescaling interval was only 275,000 years, which was far too small. The rescaling interval must be long enough that the divergence of neighboring trajectories is dominated by the exponential divergence associated with chaotic behavior rather than the power law divergence caused by the accumulation of numerical errors. In our experiment the rescaling interval should have been greater than 30 million years.

It is important to emphasize that the variational method of measuring the Lyapunov exponent has none of these problems.

Features of the orbital elements of Pluto. The largest component in the variation of h (Fig. 4) reflects the 3.7-million-year regression of the longitude of perihelion. The 27-million-year component we previously reported is clearly visible, as is the 137-million-year component. The change in density of points reflects a change in the sampling interval. For the first 450 million years of our integration we recorded the state of the system every 499,983 days (about 1,369 years) of simulated time. For the second 400 million years we sampled 16 times less frequently.

Besides the major 3.8-million-year component in the variation of the inclination of Pluto (Fig. 5) we can clearly discern the 34-million-year component we previously reported. Although there is no continuing secular decline in the inclination, there is a component with a period near 150 million years and evidence for a component with a period of approximately 600 million years.

The existence of significant orbital variations with such long periods would be quite surprising if the motion were quasiperiodic. For quasiperiodic trajectories we expect to find frequencies that are low order combinations of a few fundamental frequencies (one per degree of freedom). The natural time scale for the long-term evolution of a quasiperiodic planetary system is set by the periods of the circulation of the nodes and perihelia, which in this case are a few million years. Periods in the motion of Pluto comparable to the length of the integration have been found in all long-term integra-

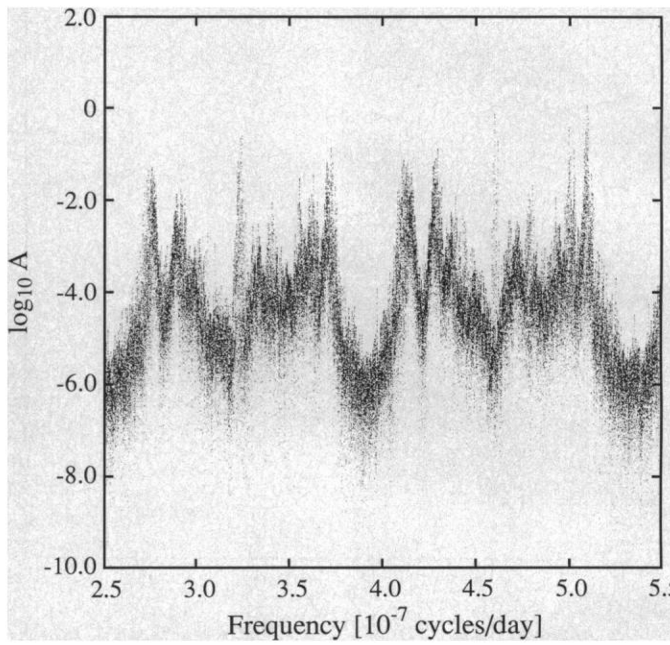

Fig. 6. A portion of the power spectrum of Pluto's h. In this graph A is the relative amplitude. There appears to be a broad-band component to the spectrum. This is consistent with the chaotic character of the motion of Pluto as indicated by the positive Lyapunov exponent.

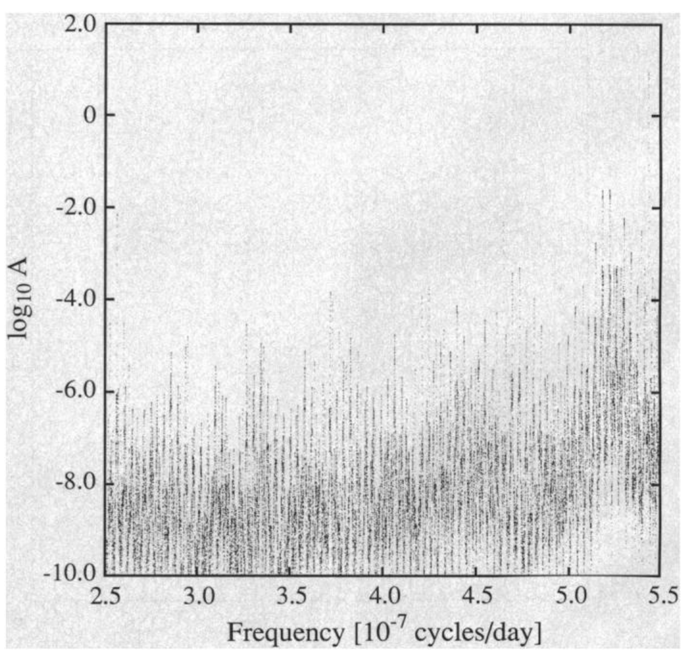

Fig. 7. A portion of the power spectrum of Neptune's h. In this graph A is the relative amplitude. The spectrum is apparently a quite complicated line spectrum. That we do not observe a broad-band component is consistent with the motion being quasiperiodic.

tions. This is consistent with the chaotic character of the motion of Pluto, as indicated by our measurement of a positive Lyapunov exponent.

Usually the measurement of a positive Lyapunov exponent provides a confirmation of what is already visible to the eye; that is, chaotic trajectories look irregular. In this case, except for the very long periods, the plots of Pluto's orbital elements do not look particularly irregular. However, the irregularity of the motion does manifest itself in the power spectra. For a quasiperiodic trajectory the power spectrum of any orbital element is composed of integral linear combinations of fundamental frequencies, where the number of fundamental frequencies is equal to the number of degrees of freedom. The power spectrum of a chaotic trajectory usually appears to have some broad-band component.

A portion of the power spectrum of Pluto's h is shown in Fig. 6. For comparison the same portion of the power spectrum of Neptune's h is shown in Fig. 7. This portion of the spectrum was chosen to avoid confusion introduced by nearby major lines. Hanning windows have been used to reduce spectral leakage; only the densely sampled part of the run was used in the computation of the Fourier transforms. The spectrum of Neptune is quite complicated but there is no evidence that it is not a line spectrum. On the other hand the spectrum of Pluto does appear to have a broad-band component. Note that both of these spectra are computed from the same integration run, by means of the same numerical methods. They are subject to the same error processes, so the differences we see are dynamical in origin. The amplitudes in both graphs are normalized in the same way, so we can see that the broad-band components in Pluto's spectrum are mostly larger than the discrete components in Neptune's spectrum.

The lack of obvious irregularity in the orbital elements of Pluto indicates that the portion of the chaotic zone in which Pluto is currently moving is rather small. Since the global structure of the chaotic zone is not known it is not possible for us to predict whether more irregular motions are likely. If the small chaotic zone in which Pluto is found connects to a larger chaotic region, relatively sudden transitions can be made to more irregular motion. This actually occurs for the motion of asteroids near the 3 : 1 Kirkwood gap (*13*).

On the other hand, the fact that the time scale for divergence is only an order of magnitude larger than the fundamental time scales of the system indicates that the chaotic behavior is robust. It is not a narrow chaotic zone associated with a high-order resonance. Even though we do not know the sensitivity of the observed chaotic behavior to the uncertainties in parameters and initial conditions, and unmodeled effects, the large Lyapunov exponent suggests that the chaotic behavior of Pluto is characteristic of a range of solar systems including the actual solar system.

Conclusions and implications of Pluto's chaotic motion. Our numerical model indicates that the motion of Pluto is chaotic. The largest Lyapunov exponent is about $10^{-7.3}$ year^{-1}. Thus the e-folding time for the divergence of trajectories is about 20 million years. It would not have been surprising to discover an instability with characteristic time of the order of the age of the solar system because such an instability would not yet have had enough time to produce apparent damage. Thus, considering the age of the solar system, 20 million years is a remarkably short time scale for exponential divergence.

The discovery of the chaotic nature of Pluto's motion makes it more difficult to draw firm conclusions about the origin of Pluto. However, the orbit of Pluto is reminiscent of the orbits of asteroids on resonant chaotic trajectories, which typically evolve to high eccentricity and inclination (*6*). This suggests that Pluto might have been formed with much lower eccentricity and inclination, as is typical of the other planets, and that it acquired its current peculiar orbit purely through deterministic chaotic dynamical processes. Of course, it is also possible that Pluto simply formed in an orbit near its current orbit.

In our experiment Pluto is a zero-mass test particle. The real Pluto has a small mass. We expect that the inclusion of the actual mass of Pluto will not change the chaotic character of the motion. If so, Pluto's irregular motion will chaotically pump the motion of the other members of the solar system and the chaotic behavior of Pluto would imply chaotic behavior of the rest of the solar system.

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