Topological Solutions in Gauge Theory and Their Computer Graphic Representation

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A diverse range of physical phenomena, both observed and hypothetical, are described by topological solutions to nonlinear gauge field theories. Computer-generated color graphic displays can provide a clear and detailed representation of some of these solutions, which might otherwise be physically unintelligible because of their mathematical complexity. Graphical representations are presented here for two topological solutions: (i) the solutions of a model that represents the filaments of quantized magnetic flux in a superconductor, and (ii) the solutions of an SO(3) gauge theory corresponding to a pair of separated magnetic monopoles. An introduction is provided to the gauge field theories giving rise to these solutions.

GAUGE THEORY IS THE GENERIC NAME FOR A CLASS OF vector field theories. It encompasses electromagnetism, the simplest gauge theory, and a group of theories that are generalizations of electromagnetism, called non-Abelian gauge theories. The study of gauge theories is of great importance in several areas of physics, particularly particle physics, as it is now generally believed that all of the fundamental interactions in nature can be explained in terms of gauge fields.

An important development of this research has been the discovery, in some gauge theories, of solutions with nontrivial boundary conditions, called "topological solutions." These are associated with a diverse range of phenomena, from quantized magnetic flux tubes in superconductivity, to instantons and magnetic monopoles in non-Abelian gauge theories. [For an introduction to the subject of topological solutions in field theory, see the book by Rajaraman (*1*) and the reprint collection edited by Rebbi and Soliani (*2*).]

Frequently, however, an obstacle to understanding these solutions is their great complexity, which stems from the fact that the field equations giving rise to topological solutions are nonlinear. Their solution often requires complicated mathematical analysis, leading to numerical results in the form of lengthy tables and graphs that permit only a limited grasp of the qualitative properties of the solutions and the phenomena they represent.

The situation may be improved greatly by modern computer-generated graphics. It is possible in many cases to generate static and dynamic representations of complex physical systems to provide intuitive insights not otherwise obtainable. Indeed, computer graphic displays are commonplace in many areas of science [see (*3*) for an introduction].

In this article we consider two phenomena described by topological solutions to gauge field theories, and present our initial attempts to create displays of these solutions. First we consider the Landau-Ginzburg model of superconductivity (*4*), whose topological solutions represent the filaments of quantized magnetic flux (strings, or vortex lines) that are observed in type II superconductors (*5*). Then, after a brief introduction to non-Abelian gauge theories, we discuss generalized magnetic monopoles, which were discovered as topological solutions of certain non-Abelian theories by 't Hooft (*6*) and Polyakov (*7*). (We remark that it is not known whether these monopoles actually exist in nature.)

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Flux Quanta in Superconductors

A simplified model to describe superconductivity was proposed in 1950 by Landau and Ginzburg (*4*). In this model, the microscopic structure of the superconductor, now understood in terms of the electron pairing theory of Bardeen, Cooper, and Schrieffer (BCS), is replaced by a complex scalar field ϕ of charge $q = 2e$ that represents the condensate of electron pairs. The field ϕ interacts with the electromagnetic potential **A**, and the static field configurations are solutions of the equations

$$\nabla \cdot \mathbf{B} = 0 \tag{1}$$

$$\nabla \times \mathbf{B} = 4\pi \mathbf{j} \tag{2}$$

$$\left(\nabla + \frac{iq}{\hbar}\mathbf{A}\right)^2 \phi = (\mu^2 + 2\lambda|\phi|^2)\phi \tag{3}$$

Here the first two equations are simply Maxwell's equations for the magnetic field **B** = curl **A**, with current density

$$\mathbf{j} = iq\hbar(\phi^*\nabla\phi - \phi\nabla\phi^*) - 2q^2|\phi|^2\mathbf{A} \tag{4}$$

associated with the flow of electron pairs (in a real material, a contribution to the current from the flow of unpaired electrons might need to be included as well). The third equation is the field equation for ϕ, where μ^2 and λ are temperature-dependent parameters related to the microscopic properties of the electron pairs and $\hbar$ is Planck's constant.

At high temperatures, the parameter μ^2 is positive, and the only solution to the equations is $\phi = 0$ and $\mathbf{B} = 0$, corresponding to an absence of electron pairs—the material behaves like a normal conductor. For a superconductor, the parameter μ^2 becomes negative below a critical temperature T_c, and the equations have a solution

$$|\phi|^2 = -\frac{\mu^2}{2\lambda} \equiv \frac{1}{2}v^2 \tag{5}$$

$$\mathbf{A} = 0 \tag{6}$$

This solution corresponds to a superconducting state with density of electron pairs proportional to $|\phi|^2$, and zero magnetic field inside the superconductor.

However, the equations with $\mu^2 < 0$ have additional solutions with interesting topological properties. These solutions have nonzero magnetic field in the superconductor, but the field is confined to a long filament (flux tube, or vortex line). The magnitude of the field decreases exponentially with distance from the axis of the filament, with characteristic length (the screening length, or London penetration depth)

$$l_s = 1/qv \tag{7}$$

(l_s is also the distance to which an external magnetic field penetrates the interior of a superconducting surface).

The density of electron pairs vanishes on the axis of the flux tube, and rises to its value in the bulk of the superconductor over a distance scale (the coherence length l_c) given by

$$l_c^2 = 1/(2\lambda v^2) = 1/(2|\mu|^2) \tag{8}$$

The total magnetic flux Φ through a cross-secton of the flux tube is quantized—it must be an integer multiple of the flux quantum

$$\Phi_0 = 2\pi\hbar/q \tag{9}$$

Here it is important to note that $q = 2e$ is the charge of the electron pair.

To understand the origin of the flux quantization, write the complex scalar field ϕ as

$$\phi = \rho e^{i\theta} \tag{10}$$

(with $\rho \geq 0$, θ real). Then the current density is given by

$$\mathbf{j} = -2q\rho^2(\hbar\nabla\theta + q\mathbf{A}) \tag{11}$$

and the magnetic flux through a closed curve C is given by

$$\Phi = \int_C \mathbf{A}\cdot d\mathbf{l} = -\frac{1}{2q^2}\int_C \frac{\mathbf{j}\cdot d\mathbf{l}}{\rho^2} - \frac{\hbar}{q}\int_C \nabla\theta\cdot d\mathbf{l} \tag{12}$$

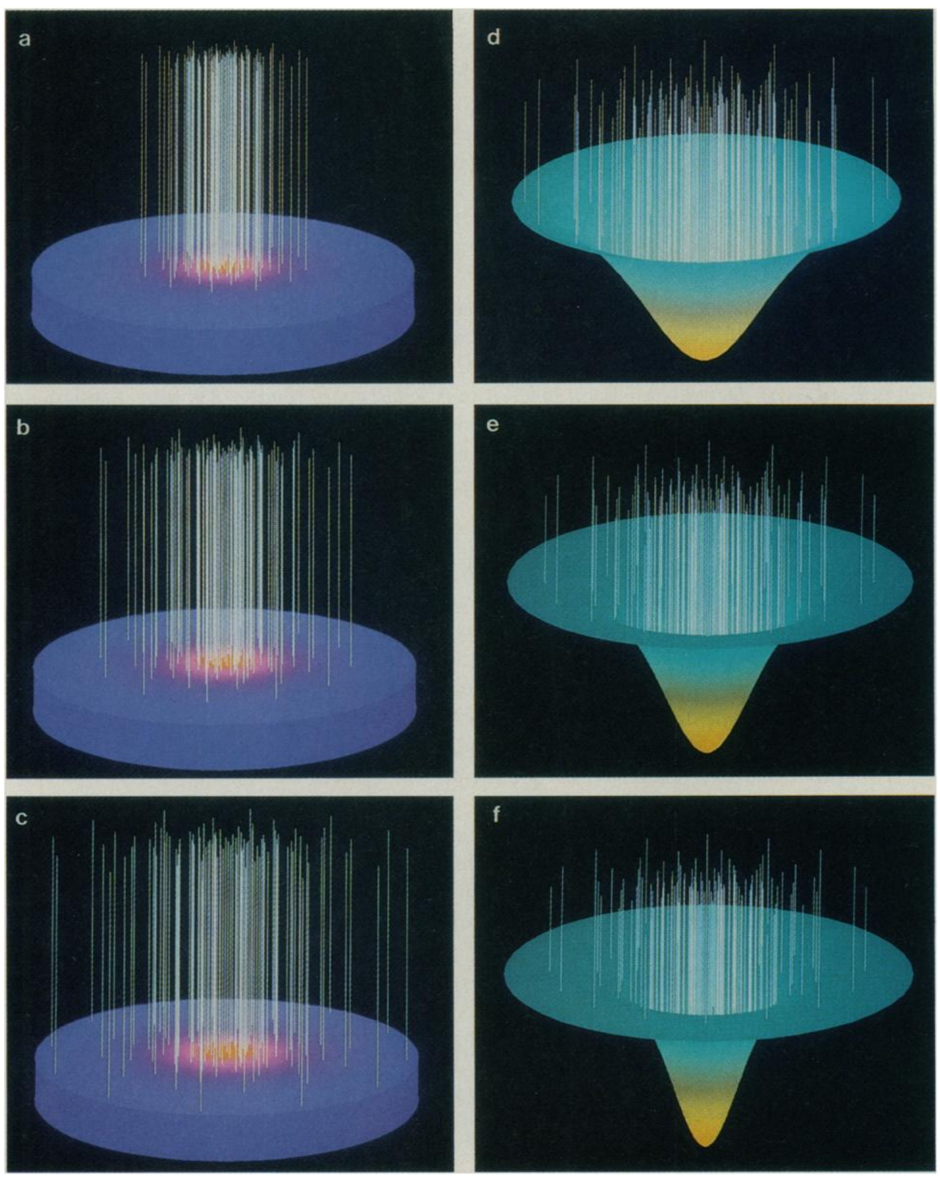

Fig. 1. Two representations of a quantized magnetic flux tube in a superconductor. The density of the vertical lines is proportional to the magnetic flux density. The color variation represents the magnitude of $|\phi|^2$, which varies from zero at the center of the flux tube to $v^2/2$ outside. In addition, the vertical displacement of the surfaces in (**d**) to (**f**) is also proportional to $|\phi|^2$. The length scale is provided by the coherence length l_c in (**a**) to (**c**), and by the screening length l_s in (d) to (f). The ratios l_s/l_c of screening length to coherence length are 0.6, 1.0, and 1.5, from top to bottom.

Now let C be a circle in a plane normal to the flux tube, with center on the axis and radius large compared to l_s and l_c. The current flow around C is negligible, so the magnetic flux is given simply by

$$\Phi = -\frac{\hbar}{q}\int_C \nabla\theta \cdot \mathbf{d}\boldsymbol{l} \tag{13}$$

This integral need not vanish, but the field ϕ must be single-valued, so that the phase angle θ must change by an integer multiple of 2π in going around the circle,

$$\int_C \nabla\theta \cdot \mathbf{d}\boldsymbol{l} = 2n\pi \tag{14}$$

($n = 0, \pm 1, \pm 2, \ldots$) and

$$\Phi = -\frac{2n\pi\hbar}{q} = -n\Phi_0 \tag{15}$$

Note that the curve C need not be a circle—any large closed loop around the flux tube in a normal plane will do, because the variation of the phase angle θ around C, as defined by Eq. 14, is unchanged by continuous deformations of C (and hence can be characterized as a topological property of the solution).

The physically interesting properties of the flux tubes are the magnetic field strength $\mathbf{B}$, the density of electron pairs, which is proportional to $|\phi|^2$, and perhaps the energy density. These quantities can be plotted on conventional graphs, but the information can also be depicted in various color graphic representations, as shown in Fig. 1.

The presence or absence of flux tubes in actual superconductors depends on the interaction between flux tubes, which in turn depends on the ratio l_s/l_c of the screening length to the coherence length. If $l_s < l_c$ (type I superconductor), magnetic flux is completely expelled from the interior of the superconductor (Meissner effect) and no flux tubes remain. If $l_s > l_c$ (type II), on the other hand, magnetic flux in the superconductor persists in flux tubes that form a triangular lattice over a cross section of the superconductor (this lattice minimizes the interaction energy of the flux tubes, which is positive when $l_s > l_c$). The total magnetic flux through a cross section of the superconductor is an integer multiple of the flux quantum Φ_0. For further discussion, see, for example, (*8, 9*).

Non-Abelian Gauge Theories

There is a long history of the study of symmetry principles in physical systems [for an introduction, see (*10*)]. Perhaps the most widely understood symmetries are the three-dimensional rotational symmetries of atoms and molecules, and the translational and rotational symmetries of crystal lattices, but more abstract symmetries play an essential role in theories of the fundamental interactions of nature.

Of special interest here are the symmetries and related conversation laws—extensions of the familiar concept of electric charge—associated with the internal structure of elementary particles. Early examples are the approximate symmetries of isospin—first proposed by Heisenberg in 1947 to account for the similarity between proton and neutron—and its extensions used to classify the rich spectrum of baryons and mesons explored at particle accelerators in the 1950s and 1960s. These symmetries are described by transformations of the internal degrees of freedom (coordinates) of elementary particles, just as rotations and translations are described by transformations of spatial coordinates.

One further level of abstraction, crucial to contemporary theories of elementary particles, is to consider internal symmetry transformations that act independently at each space-time point. These are called (local) gauge transformations, and were already known from the quantum mechanics of charged particles, where the transformations of the particle wave function have the form

$$\psi(x) \rightarrow \exp\{iq\alpha(x)\}\psi(x) \tag{16}$$

Here q is the charge of the particle, and $\alpha(x)$ is an arbitrary real function of space and time coordinates. Note that the "phase factor" $\exp\{iq\alpha\}$ has magnitude 1, so that this transformation does not alter the probability density $|\psi(x)|^2$. In fact, the requirement of invariance under the gauge transformations of Eq. 16, together with corresponding transformations of the electromagnetic potential (the "gauge field" of the theory), can be used as a starting point for the formulation of quantum electrodynamics.

The generalization of the gauge transformations of Eq. 16 to internal symmetries, with the simple phase factor replaced by a unitary matrix acting on the internal degrees of freedom, was introduced by Yang and Mills in 1954. (The group formed by the transformation is said to be non-Abelian, which means that multiplication of the matrices involved in the gauge transformations is not commutative.) However, it was not until the 1960s that theories based on invariance under these generalized gauge transformations were proposed as a unified theory of the weak and electromagnetic interactions by Glashow, Salam, and Weinberg. The now accepted theory of strong interactions (quantum chromodynamics, or QCD), which is based on invariance under gauge transformations of the "color" degree of freedom of quarks, was first constructed in the early 1970s [an elementary technical introduction is given in (*11*)].

The main obstacle to appreciating the relevance of the physically appealing gauge theories was the fact that the vector fields ("gauge fields") associated with the gauge transformations, analogous to the photon of electromagnetic theory, are apparently required to be massless (like the photon), and no such fields other than the photon had been observed. However, in a non-Abelian gauge theory, the gauge fields are no longer neutral, unlike the photon, which carries no electric charge. In QCD, it is believed that only states that are neutral under the gauge transformations of color are observable—free quarks have never been seen experimentally, and the known baryons and mesons are colorless composite states of quarks and antiquarks. Hence free gauge fields ("gluons") are expected to be unobservable—like quarks, they are seen only indirectly in composite state hadrons (known as "glueballs") or as "jets" of hadrons produced at high-energy accelerators.

In the unified theory of weak and electromagnetic interactions, the requirement of massless gauge fields is evaded by a mechanism (generally known as the Higgs mechanism) that can generate massive gauge fields. In this theory, there is a scalar field ϕ (the Higgs scalar) that has a nonzero value in the state of lowest energy (the vacuum), just as the scalar field in the Landau-Ginzburg model of superconductivity has a nonzero value given by Eq. 5. This vacuum state no longer has the symmetry of the original theory (a useful analogy is the ground state of a ferromagnet—even though the interactions between atoms are spherically symmetric, the ground state is not, since the direction of the magnetization defines a special direction in space). In these circumstances, the excitations of the vacuum corresponding to gauge fields that do not represent symmetries of the vacuum appear as massive vector particles. In the electroweak theory, these are identified with the W and Z particles discovered at CERN in 1982.

Although the scalar field has been introduced to generate a mass for the W and Z particles, it is not well understood. Indeed, many of the outstanding problems of elementary particle physics center on the properties of the Higgs scalar. Where is the particle (or particles) corresponding to the field? What are the properties of these particles? What is the origin of the nonzero value of the field in the vacuum of the theory?

Magnetic Monopoles in Non-Abelian Gauge Theories

Maxwell's equations for the electromagnetic fields **E** and **B** have a symmetry between electric and magnetic fields in the absence of sources. However, the sources do not display this symmetry—electric fields are generated by electric charges, whereas magnetic fields are generated by currents associated with moving electric charges. Although this picture is in complete agreement with present experimental knowledge, it is useful to contemplate the consequences of the existence of isolated magnetic charges, or monopoles.

An important result, derived by Dirac in a pair of classic papers (*12*), is that the fundamental units g_e and g_m of electric and magnetic charge, respectively, are related by

$$g_e g_m = n/2 \tag{17}$$

where n is an integer (throughout this section we have set $c = \hbar = 1$). The topological origin of this relation as a condition for the consistency of the gauge transformations of Eq. 16 in the presence of magnetic charges was later clarified by Yang (*13*). One attractive consequence of Eq. 17 is that it provides a natural explanation of the observed quantization of electric charges in units of the electron charge e (or, more recently, the quark charge $e/3$).

Although magnetic monopoles are allowed within the framework of standard electromagnetic theory, they are not compulsory, and it is interesting to consider what happens when electromagnetism is embedded into a non-Abelian gauge theory. In 1974, 't Hooft (*6*) and Polyakov (*7*) noted that in a theory with gauge group SO(3), corresponding to rotations in a three-dimensional internal space (with the same structure as rotations of ordinary three-dimensional space), there are static (time-independent) solutions to the field equations that have many of the properties associated with magnetic monopoles. In particular, there is a localized source from which emerges a long-range magnetic field characteristic of a monopole. However, the fields within the source are non-singular, so that the total energy in the configuration is finite and calculable, and the mass of the associated particle can be predicted (unlike the masses of quarks and leptons in the standard model, which must be introduced as parameters to be determined from experiment).

To describe these static solutions, we introduce the fields of the theory. These consist of a triplet of scalar fields $\vec{\phi} = (\phi^a)$ and a triplet of vector potentials $\vec{\mathbf{A}} = (\mathbf{A}^a)$. Here $a = 1, 2, 3$ is an index that labels the components of a vector in the three-dimensional internal symmetry space (to be called isospin space for brevity). The arrow denotes a vector in isospin space; boldface type a vector in ordinary space.

The magnetic fields $\vec{\mathbf{B}} = (\mathbf{B}^a)$ are related to the vector potentials by

$$\mathbf{B}^a = \nabla \times \mathbf{A}^a - g\mathbf{A}^b \times \mathbf{A}^c \tag{18}$$

Here ∇ is the usual derivative operator; (a, b, c) denotes a cyclic permutation of (1, 2, 3), and g is a parameter that corresponds to the electric charge in electromagnetic theory.

The field equations for static fields can then be written as

$$\mathbf{D} \times \mathbf{B}^a = g[(\mathbf{D}\phi^b)\phi^c - (\mathbf{D}\phi^c)\phi^b] \tag{19}$$

$$\mathbf{D} \cdot \mathbf{D}\vec{\phi} = \lambda(\vec{\phi} \cdot \vec{\phi} - v^2)\vec{\phi} \tag{20}$$

where λ and v are parameters associated with the scalar fields, analogous to those introduced in Eqs. 3 and 5, and **D** is an operator that acts on a vector $\vec{V} = (V^a)$ in isospin space according to

$$\mathbf{D}V^a = \nabla V^a - g(\mathbf{A}^b V^c - \mathbf{A}^c V^b) \tag{21}$$

The extra terms proportional to g in Eqs. 18 and 21 are required so that $\mathbf{B}^a$ and $\mathbf{D}V^a$ will actually be components of a vector in isospin space.

Finite energy solutions of Eqs. 19 and 20 must have $\vec{\phi} \cdot \vec{\phi} \to v^2$ at spatial infinity. The simplest solution, corresponding to the lowest energy (vacuum) state, has $\vec{\phi} = \vec{v}$, a constant isospin vector with $\vec{v} \cdot \vec{v} = v^2$. The direction of $\vec{v}$ is undetermined, but two solutions with different directions of $\vec{v}$ are related by a (gauge) rotation in isospin space, and hence are to be considered equivalent. (In the analogous case of a ferromagnet described above, the direction of the magnetization in the ground state is well defined, but its orientation in space is arbitrary.) Two of the three gauge fields acquire a mass by the Higgs mechanism, but one massless gauge field remains, because the vacuum still has a symmetry corresponding to rotations about the direction in isospin space defined by the vector $\vec{v}$. We can identify this massless field with the usual electromagnetic field.

There are also static solutions with $\vec{\phi} \cdot \vec{\phi} \to v^2$ at infinity in which the direction of the vector $\vec{\phi}$ in isospin space varies smoothly from point to point on the "sphere at infinity" (S_∞) in ordinary space. In the simplest such solution, the isospin direction of $\vec{\phi}$ at a point on S_∞ can be identified with the direction of the radius vector from some origin to the point on S_∞. This solution is the 't Hooft–Polyakov monopole (*6, 7*)—application of Gauss' Law to the associated magnetic field shows that the total magnetic charge of the configuration is

$$m = 1/g \tag{22}$$

More generally, as points in space cover the sphere S_∞, $\vec{\phi}$ must cover the sphere $\vec{\phi} \cdot \vec{\phi} = v^2$ in isospin space an integer number n times—this is the analogue of the phase condition Eq. 14. The magnetic charge of the corresponding solution is then given by

$$m = n/g \tag{23}$$

For a deeper discussion of these points, see the comprehensive review by Goddard and Olive (*14*).

These general properties do not guarantee the existence of solutions with $n > 1$. Indeed, static (time-independent) solutions exist only in the special limit $\lambda \to 0$, while retaining the boundary condition $\vec{\phi} \cdot \vec{\phi} \to v^2$ at infinity. This is called the Prasad-Sommerfield limit, after those who obtained the analytic single monopole ($n = 1$) solution in this limit (*15*). The existence of static multi-monopole ($n > 1$) solutions in the Prasad-Sommerfield limit is a consequence of the fact that the scalar field $\vec{\phi}$ becomes massless as $\lambda \to 0$. It turns out that in this case the long-range magnetic repulsion between monopoles is exactly balanced by a long-range attraction due to the interaction with the scalar field. Very little is known about the (necessarily) time-dependent solutions with $\lambda \neq 0$. Nonetheless, the study of the static solutions provides a useful starting point for the study of the properties of such monopoles.

In the Prasad-Sommerfield limit, the field equations Eqs. 19 and 20 simplify to first-order form

$$\vec{\mathbf{B}} = -\mathbf{D}\vec{\phi} \tag{24}$$

Equation 24 still represents a set of coupled nonlinear partial differential equations whose solution has involved the use of mathematical techniques that were only developed in the 1970s [see (*2*) for a representative sample]. What has finally emerged, after many important contributions not chronicled here, is a complete set of formal solutions corresponding to a collection of n monopoles at arbitrary positions (*16–19*). However, even the simplest such solution, corresponding to two monopoles ($n = 2$), is extremely complicated and as such physically unenlightening. Numerical methods must be used to evaluate the solution, and computer graphics are certainly a great asset in appreciating its very interesting properties.

The pictures in Fig. 2 are plotted from the two-monopole solution derived by Forgács *et al.* (*19*) [gauge group SO(3), $\lambda = 0$]. To our knowledge this is the first time that the two-monopole system has been studied graphically in such detail.

Figure 2, a to e, shows the total energy density (that is, the sum of the magnetic and Higgs energy densities) for various monopole separations. When the monopoles are far apart (Fig. 2a) they look like a pair of isolated Prasad-Sommerfield monopoles. As they approach one another, however, the energy density evolves in a curious way (Fig. 2, b to e). It is not symmetrically distributed about the axis joining the monopoles (unlike the electric field of two stationary electrons or the magnetic field of a bar magnet). Neither is it spherically symmetric when they are coincident (Fig. 2e). In the latter case it is symmetric about an axis, but that axis is perpendicular to the line of approach of the monopoles. This is in accordance with a general result that $n > 1$ solutions are not spherically symmetric and are axially symmetric only if the magnetic sources are coincident (*20*).

Figure 2f plots the squared-magnitude of the scalar field, $|\vec{\phi}|^2$, at an intermediate monopole separation. This quantity tends to v^2 at spatial infinity.

We should remark that no monopoles have actually been observed; neither is their existence required in the standard QCD-electroweak theory. However, they are predicted to exist in more ambitious "grand unified" theories that incorporate the color and electroweak gauge groups into a larger gauge group, although the density of such monopoles in the universe may be so low as to make their detection quite unlikely.

Concluding Remarks

We have demonstrated, by means of two specific examples, that computer-generated color graphic displays are a useful tool in understanding the qualitative nature of complicated solutions from gauge theory. Indeed, such representations of the two-monopole system may well provide new insights into this and mathematically similar sytems.

We close with some remarks concerning a new trend in computer architecture that promises to make such pictures much quicker and cheaper to produce, and hence more widely available. It is the use of parallel processing, which is often well suited to this type of problem.

Consider, for example, plotting a quantity whose dependence on a set of parameters is given by a known analytic (but possibly complicated) formula. Then the calculation of that quantity throughout some region of parameter space can easily be "farmed out" to a network of processors, each running the same code and working on a portion of the total region of interest. This technique

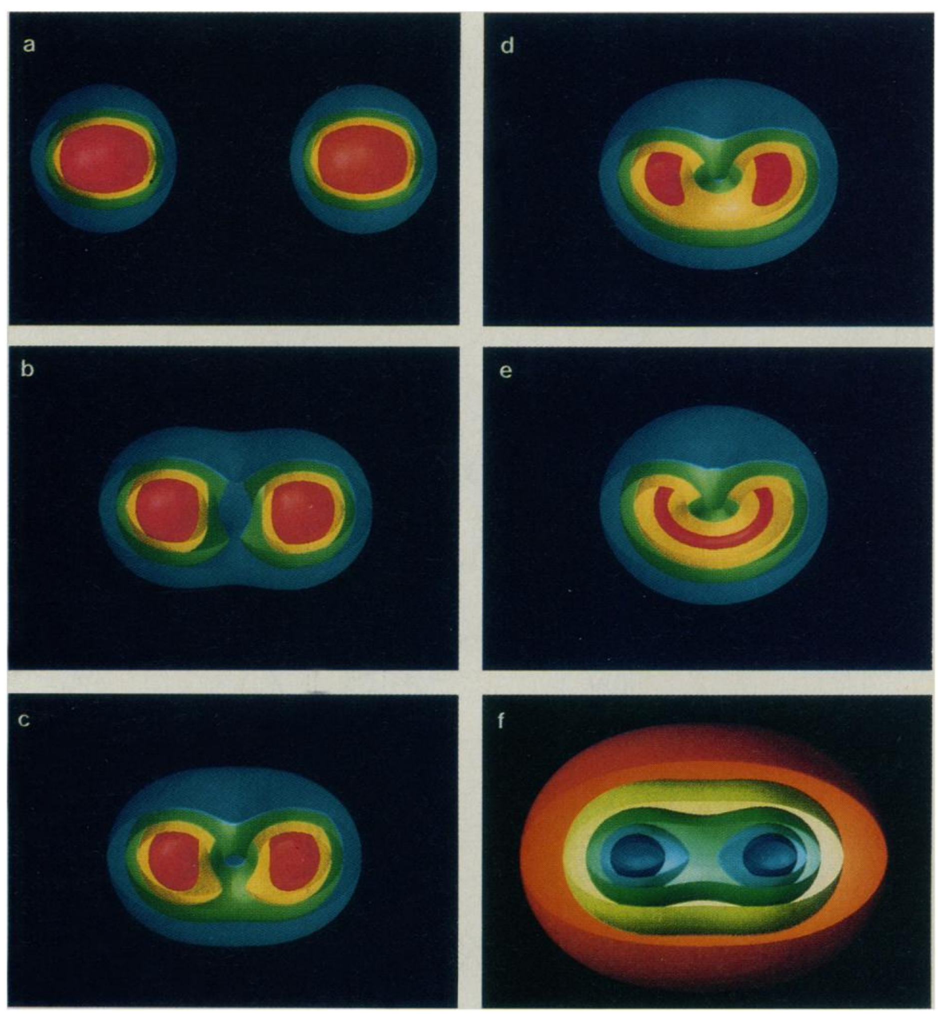

Fig. 2. A pair of static SO(3) magnetic monopoles at various separations. (**a** to **e**) Surfaces of constant total energy density (magnetic + Higgs energy density). The separations, measured between the zeroes of the Higgs field, are 10, 4, 3, 2, and 0, respectively (in units of e/M, e = electric charge, M = gauge particle mass). The energy density values are: blue = 1, green = 2, yellow = 3, red = 4 (in units of $0.02\ M^4/e^5$). The energy is not symmetrically distributed about the axis joining the monopoles. Neither is it spherically symmetric when the monopoles are coincident, although in this case it is symmetric about an axis perpendicular to their line of approach. (**f**) Surfaces of constant Higgs field squared, $|\phi|^2$, for a monopole separation of 4. The $|\phi|^2$ values are: dark blue = 0.4, light blue = 0.2, green = 0.1, yellow = 0.05, red = 0.025 (with $v^2 = 1$ in Eq. 20).

was used in the present study to generate the two-monopole data for plotting. The calculation was performed on a "Meiko Computing Surface" containing 16 Inmos T414 transputers. In principle the problem could be farmed out to a processor array of arbitrary size with proportionate speed up.

Similarly, the actual plotting can be farmed out, each processor working on a subset of the complete picture. Although the displays presented here were generated using conventional computers, an algorithm for three-dimensional solid modeling on a processor array is currently under development. With this technology it should be possible to plot such pictures in seconds rather than hours.

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