Articles

The Science of Patterns

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The rapid growth of computing and applications has helped cross-fertilize the mathematical sciences, yielding an unprecedented abundance of new methods, theories, and models. Examples from statistical science, core mathematics, and applied mathematics illustrate these changes, which have both broadened and enriched the relation between mathematics and science. No longer just the study of number and space, mathematical science has become the science of patterns, with theory built on relations among patterns and on applications derived from the fit between pattern and observation.

Modern mathematics just marked its 300th birthday. The publication in 1687 of Newton's *Principia Mathematica* established mathematics as the methodological paradigm of theoretical science. Newton perceived patterns in the accumulated astronomical data of his time; he abstracted from these patterns certain general principles (whence *Principia*); then he used these principles to deduce patterns both known and unknown in the behavior of planetary bodies. His was a science of patterns—rooted in data, supported by deduction, confirmed by observation.

By the end of the 19th century, Newton's creation had flowered magnificently, producing unprecedented intellectual blossoms. European giants such as Euler, Lagrange, and Weierstrass had elaborated and refined the calculus, establishing the foundations for modern analysis. James Clerk Maxwell used Newton's derivatives to write the laws of electromagnetism, and Georg Bernhard Riemann applied differentials to geometry in apt (albeit unintentional) preparation for Albert Einstein, who soon would discover in Riemannian geometry the key to a general theory of gravitation.

At the same time, on a separate continent, the people of the United States were beginning their second century without mathematical or scientific giants in their midst. Yet in 1888, two centuries after *Principia*, a few far-sighted individuals founded what is now the American Mathematical Society, thereby setting in motion a process that created the world's strongest environment for mathematics research. In recognition of this anniversary, American mathematics is now celebrating its 100th birthday.

Forces for Change

Many educated persons, especially scientists and engineers, harbor an image of mathematics as akin to a tree of knowledge: formulas, theorems, and results hang like ripe fruits to be plucked by passing scientists to nourish their theories. Mathematicians, in contrast, see their field as a rapidly growing rain forest, nourished and shaped by forces outside mathematics while contributing to human civilization a rich and ever-changing variety of intellectual flora and fauna. These differences in perception are due primarily to the steep and harsh terrain of abstract language that separates the mathematical rain forest from the domain of ordinary human activity.

The dense jungle of mathematics has been nourished for millennia by challenges of practical applications. In recent years, computers have amplified the impact of applications; together, computation and applications have swept like a cyclone across the terrain of mathematics. Forces unleashed by the interaction of these intellectual storms have changed forever—and for the better—the morphology of mathematics. In their wake have emerged new openings that link diverse parts of the mathematical forest, making possible cross-fertilization of isolated parts that has immeasurably strengthened the whole.

Throughout the 20th century, mathematics has grown rapidly on many fronts. The classical core has remained rooted in the Newtonian mathematics of analysis, a synthesis of algebra and geometry applied to the study of how things change. But even as this core has expanded under explosive post–World War II growth, it has been supplemented by major developments in other mathematical sciences—in number theory, logic, statistics, operations research, probability, computation, topology, and combinatorics, in addition to algebra, geometry, and analysis.

In each of these subdisciplines, applications parallel theory. Even the most esoteric and abstract parts of mathematics—number theory and logic, for example—are now used routinely in applications (for example, in computer science and cryptography). Fifty years ago G. H. Hardy could boast of number theory as the most pure and least useful part of mathematics (*1*); today number theory is studied as an essential prerequisite to many applications of coding, including data transmission from remote satellites, protection of financial records, and efficient algorithms for computation.

In 1960, at a time when theoretical physics was still the central jewel in the crown of applied mathematics, Eugene Wigner wrote about the "unreasonable effectiveness" of mathematics in the natural sciences: "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve" (*2*, p. 14). Indeed, theoretical physics has continued to adopt (and occasionally invent) increasingly abstract mathematical models as the logical foundation for current theories: Lie groups and gauge theories, exotic expressions of symmetry, have joined fermions and baryons as fundamental tools in the physicist's search for a unified theory of both microscopic and macroscopic forces of nature.

During this same period, however, striking applications of mathematics emerged across the entire landscape of natural, behavioral, and social science. Moreover, applications of one part of mathematics to another—of geometry to analysis, of probability to number theory—provide renewed evidence of the fundamental unity of mathematics. Despite the ubiquity of connections among problems in science and mathematics, the discovery of new links retains a surprising degree of unpredictability and serendipity. Whether

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planned or unplanned, the cross-fertilization between science and mathematics in problems, theories, concepts, and paradigms has never been greater than it is now, in the last quarter of the 20th century. In 1988 one can say with some justification that the effectiveness of mathematics is even more "unreasonable" than ever before.

Paralleling the growing power of applications of mathematics has been the extraordinary impact of computing. It is ironic but indisputable that computers were made possible by application of abstract theories of mathematicians such as Boole, Cantor, Turing, and von Neumann, theories that just a few decades ago were widely derided by critics of the "new math" as wild abstractions irrelevant to practical purposes. It is doubly ironic that the computer is now the most powerful force changing the nature of mathematics. Even mathematicians who never use computers may frequently devote their entire research careers to problems generated by the presence of computers. Across all parts of mathematics, computers have posed new problems for research, provided new tools to solve old problems, and introduced new research strategies.

Although the public often views computers as a replacement for mathematics, each is in reality a power tool for the other. Indeed, just as computers provide new opportunities for mathematics, so also mathematics makes computers so incredibly effective. Mathematics provides abstract models for natural phenomena, as well as algorithms for implementing these models in computer languages. Applications, computers, and mathematics form a tightly coupled system yielding results never before possible and ideas never before imagined.

The Mathematical Sciences

Rapid growth in the nature and applications of mathematics means that the Newtonian core—calculus, analysis, and differential equations—is now just one part of a more diverse mathematical landscape. Yet most scientists have explored only this original territory, because that is all that was included in their curriculum in high school, college, and graduate school. With the exception of statistics, an old science widely used across all disciplines that has become largely mathematical during the 20th century, the narrow Newtonian legacy of analysis is the principal connection between practicing scientists and the broad mathematical foundations of their disciplines. The dramatic changes in the mathematical sciences of the last quarter century are largely invisible to those outside the small community of research mathematicians.

Today's mathematical sciences, like yesterday's Gaul, can be divided into three parts of roughly comparable size: statistical science, core mathematics, and applied mathematics. Each of these three major areas is led (in the United States) by a few thousand active researchers and receives approximately $50 million in federal research support annually. Although the boundaries between these parts overlap considerably, each province has an identifiable character that corresponds well with the three stages of the mathematical paradigm established by Newton: data, deduction, and observation.

Statistical science investigates problems associated with uncertainty in the collection, analysis, and interpretation of data. Its tools are probability and inference; its territory includes stochastic modeling, statistical inference, decision theory, and experimental design. Statistical science influences policy in agriculture, politics, economics, medicine, law, science, and engineering. Advances in instrumentation and communication (to gather and transmit data) have posed new challenges to statistics, leading to rapid growth in new methods and new applications.

Core mathematics investigates properties of number and space, ideas rooted in antiquity. Its tools are abstraction and deduction; its edifices include functions, equations, operators, and infinite-dimensional spaces. Within core mathematics are found the traditional subjects of number theory, algebra, geometry, analysis, and topology. After a half-century of explosive specialized growth, core mathematics is experiencing a renaissance of renewed integrity based on the unexpected but welcome discovery of deep links among its various components.

Applied mathematics fits mathematical methods to the observations and theories of science. It is a principal conduit for scientific ideas to stimulate mathematical innovation and for mathematical tools to solve scientific problems. Traditional methods of applied mathematics include differential equations, numerical computation, control theory, and dynamical systems; such traditional methods are today being applied in major new areas of applications, including combustion, turbulence, optimization, physiology, and epidemiology. In addition, new tools from game theory, decision science, and discrete mathematics are being applied to the human sciences where choices, decisions, and coalitions rather than continuous change are the apt metaphors for description and prediction.

All attempts to divide mathematics into parts are necessarily artificial and perpetually in flux: statistical science, core mathematics, and applied mathematics represent just one of many possible structures that may help one understand the whole. These divisions do not represent intrinsic differences in the nature of the discipline so much as differences in style, purpose, and history; they may more aptly describe types of mathematicians than types of mathematics. Others have attempted to portray the nature of modern mathematics in somewhat different terms [see, for example, (*3–5*)]. What is important about these labels is that they help focus attention on certain characteristics of the mathematical sciences, not that they themselves represent inherent or essential compartments.

One feature that is inherent in mathematics and is essential to understand its role in science is this: today's mathematical sciences are very different from what they were a quarter-century ago when most of today's scientists last studied mathematics. Computers, applications, and cross-fertilization have combined to transform the mathematical sciences into an extraordinarily diverse and powerful collection of tools for science. Without even asking permission, mathematicians have quite literally rebuilt the foundations of science. The work is not finished, but its new shape is sufficiently visible that all who use mathematics should take the time to explore its new features.

Statistical Sciences

Both computers and statistics deal with data: what computers record, transform, and manipulate, statisticians interpret, summarize, and display. This confluence of problem source with problem solver has radically transformed (some would say restored) statistical science to a data-intensive discipline. Phenomena described by traditional statistical distributions (normal, Poisson, and so forth) represent only a tiny part of the enormous quantity of data captured by computers all over the world. Here are three examples of what the dominance of data has done in the statistical sciences.

Spatial statistics. The increasing use of electronic scanning devices (for example, in tomography, airborne reconnaissance, and environmental monitoring) has produced an urgent need for sophisticated analysis of data with inherent spatial structure. Image enhancement, the visual clarification of blurred images, is the most common application. Other important tasks include the visual representation of data to enable an observer to detect hidden patterns and the statistical compression of data in real time to permit efficient storage and subsequent analysis without loss of important information.

Research in spatial statistics utilizes a wide variety of mathematical, statistical, and computational techniques. Problems of separating signal from noise borrow techniques from engineering; ill-posed scattering problems use methods of numerical linear algebra; smoothing of data requires statistical techniques of regularization. And underlying all this is the inherent geometry of the problem, which in many cases is dynamic and nonlinear.

Nonparametric modeling. The most fully developed part of traditional statistics involves models based on assumed distributions determined by a small number of continuously varying parameters. Yet most real data (for example, census information or satellite imaging) are a mixture of variables that are partly parametric and partly nonparametric or partly continuous and partly discrete. Without plausible a priori assumptions concerning the distribution of nonparametric data, traditional statistical methods were often unreliable—although still frequently used. Now, however, large data sets of mixed parametric and nonparametric variables make it possible for statisticians to use computationally intensive methods to estimate statistics (for example, regression coefficients) with reliable error bounds for nonparametric variables. (Of course, the computer has made such methods not only possible but necessary by enabling science to inundate us with massive data sets.)

Bootstrap and jackknife statistics. Many applications of statistics (notably survey analysis and clinical data from innovative medical protocols) involve small data sets from which one would like to infer meaningful ("significant") patterns. Bradley Efron at Stanford University has pioneered an innovative method of using limited data to generate more data with the same statistical characteristics (whence "bootstrap" methods).

In particular, bootstrap methods use computational methods to resample the given data repeatedly in order to generate millions of similar possible data sets, which yield accurate approximations to various complex statistics. By comparing the value of these statistics for the given sample with the distribution obtained by repeated resampling, one can determine whether the observed values are significant (*6*). Jackknife methods are related to bootstrap techniques, but the way they reduce bias in the statistical procedures is to repeatedly slice away part of the data.

Core Mathematics

The forces that impinge on mathematics—primarily applications, computers, and cross-fertilization—influence the core ("pure") parts of the subject in profound ways. To illustrate the nature of the changes, I will use two rather different areas of impact, computation and geometry, as widely separated mileposts along a vast continuum of mathematics.

Core mathematics has changed under the influence of computers as much as the more applied areas of the mathematical sciences, but in different ways. Most noticeable is the shift in research interests to questions motivated by computation. But computers have also changed the way conjectures are invented and tested, the way proofs are discovered, and—in an increasing number of cases—the nature of proof itself.

The archetype event in computer-assisted mathematics was the 1976 proof of the century-old four-color conjecture, which was based on a computer analysis of thousands of reduction patterns to bridge a gap between mathematical theory and human analysis of cases. This event shook the very epistemology of mathematics (*7, 8*). Ten years later, in 1986, the International Congress of Mathematicians set forth the state of worldwide mathematics at a 10-day conference at the University of California at Berkeley. Of the 16 plenary lectures surveying the current state of mathematics, more than half were on topics linked in some way with computation. Here are some examples.

Bieberbach conjecture. Louis de Branges of Purdue University discussed his proof of the 70-year-old Bieberbach conjecture concerning the size of coefficients in the power series expansions of certain analytic functions of a complex variable (*9*). At a crucial stage in the proof, de Branges had reduced the entire argument to verifying an inequality between two polynomials: this was done by computer to a sufficient degree to provide convincing circumstantial evidence of the validity of this line of argument. The final link in the formal argument was supplied by a theoretical proof of this inequality, actually known and proved in the theory of special functions long before de Branges needed it.

Factoring integers. Henrick W. Lenstra of the University of Amsterdam applied algebraic geometry to one of the oldest problems in mathematics—how to find the prime factors of an integer (*10*). Integer factorization moved from a backwater to high priority in mathematics simply because of its application in computer-based cryptography: a code based on the product of two large prime numbers cannot be broken with current algorithms because there is no known efficient method of recovering the two factors from knowledge only of the product.

Lenstra attacked the problem of factoring by using elliptic curves, the set of zeroes in a suitable projective plane of cubic polynomial equations in three variables. Points on these curves form an Abelian group, whose properties lead directly to the fastest algorithm yet discovered for factoring large numbers. Similar analyses are being carried out for factoring polynomials efficiently (*11*). Elliptic curves, the primary tool in these new algorithms, are the central objects involved in Gerd Faltings's 1983 proof of the Mordell conjecture (*12*), surely one of this decade's most stunning mathematical achievements.

Computational complexity. Arnold Schönhage of the University of Tübingen applied the theory of computational complexity—the analysis of the inherent difficulty in solving problems—to the basic task of solving equations, beginning with the simplest, $ax = b$, for ordinary numbers (*13*). The traditional solution, division, has not been of interest to mathematicians since the Middle Ages, yet now it is a subject of intense research. Only recently has it been proved that the known methods of performing division of complex numbers is the best way possible—in the sense that no method can involve fewer real arithmetical operations. The methods used in this fundamental analysis of arithmetic presage similar analysis for the entire range of computer algorithms, enabling mathematicians to determine limits of computational feasibility as well as areas ripe for further improvement.

Iterated maps. Stephen Smale of the University of California at Berkeley examined the classical problem of finding zeroes of polynomials in the very modern context of iterated maps and computational complexity theory. The prototypical example is Newton's method for finding a zero of a function $f(x)$:

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

In the pre-computer style, this method was applied one point at a time, beginning with a reasonably good guess so that the sequence of points converged rapidly to a zero of the function.

Smale studied Newton's method globally, looking at the mapping of the entire complex plane onto itself generated by Newton's method for a particular polynomial (*14*). The distribution of points that eventually map to zero is an example of a "fractal" set that displays many of the chaotic properties typical of turbulence. Smale's analysis applied global iteration to the simplex method of solving linear programming problems, leading to a proof of the empirical fact that, on average, the simplex method behaves very well indeed. His methods also lead directly to new models for chaos that have extensive applications in physics and chemistry (*15, 16*).

Geometry

If computers typify the modern era of mathematics, geometry epitomizes its classical roots. Historically, geometry, the study of space, has been one of the major pillars of core mathematics. For various reasons, its role in the mathematics curriculum has declined over the past 20 years, so that even those with university degrees in mathematics often have little acquaintance with geometry beyond an archaic and typically rigid encounter with Euclidean proofs in high school geometry. In sharp contrast to this curricular decline is the renaissance of geometry in research mathematics. In a very real sense, geometry is once again playing a central role on the stage of mathematics, much as it did in the Greek period.

A principal actor in the study in modern geometry is a "manifold," a term used by geometers to describe surfaces and spaces that are locally like Euclidean space. Manifolds form the natural locus for solutions to differential equations, and in turn their geometry imposes structure on the analytic nature of these solutions. Thus manifolds are of importance not just to geometry but to all parts of classical analysis.

Two of the three 1986 Fields Medals—the "Nobel prizes" of mathematics—went to Michael Freedman of the University of California at San Diego and Simon Donaldson of Oxford University for work in the geometry of four-dimensional manifolds (*17*). Freedman showed that the Poincaré conjecture concerning spheres is true for four-dimensional manifolds, thereby resolving the next-to-last case of this 80-year-old conjecture. (Only dimension three remains unsettled.) Freedman's methods showed that the topological classification of four-dimensional manifolds mimics the algebraic classification of quadratic forms.

Donaldson used the Yang-Mills field equations of mathematical physics, themselves generalizations of Maxwell's equations, to study instantons in four-dimensional space, thereby reversing tradition by applying methods from physics to the understanding of pure mathematics. By exploiting those properties of differential equations that reflect the wave-particle duality of matter, Donaldson was able to develop an entirely new approach to the study of fundamental problems of geometry.

One consequence of Freedman's and Donaldson's work was the discovery that in four dimensions there are differentiable manifolds that are topologically but not differentiably equivalent to the standard Euclidean four-dimensional space. Such "exotic" spaces are unique to dimension four, which also happens to be the natural domain of the space-time continuum of our physical world. Whether these unique properties are accidental or significant is something that will require much further investigation. Indeed, the geometrical theory created by this work has already led to applications in string theory, thereby feeding back into physics spinoff benefits of ideas originally borrowed from physics.

Computer graphics. Geometry and computers intersect in one of the most lively and attractive interstices of the mathematical sciences: computer graphics. Although most well-known uses of computer graphics are in areas of applied mathematics, visual techniques are making a real impact in traditional core mathematics as well as in the teaching of mathematics at all levels.

To produce realistic graphics on a computer requires considerable theoretical interaction among geometrical representation, algebraic encoding, and computer algorithms. In return, computer graphics methods have provided crucial assistance in many mathematics problems: new minimal surfaces have been found with the aid of computer graphics, and the visual displays of iterative maps (in the widely known "fractal" pictures) make visible patterns that would never have been noticed by analytic means alone. Fractal patterns have provided an apt description of a large class of physical phenomena, ranging from fracturing of glass to texture in surfaces (*18–20*).

Geometric computing is beginning to prove very useful also in core areas of mathematics remote from geometry, because supercomputers can calculate and display in visual form roots of equations and other mathematical objects. This enables mathematicians for the first time to "see" the content of the abstract theorems they prove and thereby to make new conjectures suggested by the eye rather than by the mind. (In recognition of this new frontier, several of the world's leading geometers have recently received funding to establish a Geometry Supercomputer Project to carry out research in this area in both the United States and Europe.)

Applied Mathematics

Applied mathematics is distinguished from core mathematics not so much by content or method as by objective: in applied mathematics, value is measured by the degree to which new methods improve scientific understanding or technological applications.

The roots of the scientific revolution lie in the introduction by Galileo of empirical methods to replace the speculative explanations of classical Greek natural philosophy. Newton introduced theoretical science by showing that empirical data can be explained by mathematical results deduced from basic axioms. In our time, John von Neumann pioneered the computational paradigm in which the results of theoretical science are used to simulate reality on a modern computer. As a consequence, computational methods now pervade all aspects of applied mathematics (*21, 22*).

Necessity is the mother of invention, in mathematics as in life. Because the needs of science stimulate the growth of mathematics, as science expands and grows so does mathematics. The consequence has been an explosive growth in the nature and range of applied mathematics. Four very different areas will serve to illustrate both the diversity and the innovation of present research.

Biological sciences. Nothing better illustrates the potential for mathematics in the biological sciences than the many traces of mathematics behind the Nobel prizes (*23*). For example, the 1979 Nobel Prize in medicine was awarded to Allan Cormack for his application of the Radon transform, a well-known technique from advanced classical analysis, to the development of tomography and computer-assisted tomography (CAT) scanners. One of the recipients of the 1985 Nobel Prize in chemistry was biophysicist Herbert Hauptman, who took his Ph.D. in mathematics and who is president of the Medical Foundation of Buffalo. Hauptman was cited for fundamental work in Fourier analysis pertaining to x-ray crystallography.

Indeed, recent research in the mathematical sciences suggests dramatically increased potential for fundamental advances in the life sciences with methods that depend heavily on mathematical and computer models. Structural biologists have become genetic engineers, capturing the geometry of complex macromolecules in supercomputers and then simulating interaction with other molecules in their search for biologically active agents. Using these computational methods, biologists can portray on a computer screen the geometry of a cold virus—an intricate polyhedral shape of uncommon beauty and fascinating geometric features—and search its surface for molecular footholds on which to secure their biological assault.

Geneticists are beginning the monumental effort to map the entire human genome, an enterprise that requires expertise in statistics, combinatorics, artificial intelligence, and data management to organize billions of bits of information coming from all over the world. Ecologists, the first mathematical biologists, continue to use the extensive theories of population dynamics to predict the behavior and interaction of species, taking into account such realistic complications as sex-specific mortality, reproductive biology, and predator-prey data. Neurologists now use the theory of graphs to model networks of nerves in the body and the neural tangle in the brain. And, finally, physiologists use contemporary algorithms applied to 19th-century equations of fluid dynamics to determine such things as the effects of turbulence in the blood caused by swollen heart valves or clumps of cholesterol.

Control theory. One of the most widespread uses of mathematical techniques is in the control of systems; applications range from quality control on an assembly line to flight control of a high-performance aircraft. Control theory is one of the many mathematical products of World War II, and until recently it has been dominated by the original paradigm of a single input–single output system that can be represented mathematically as a single-valued function of one variable. Such models are sufficiently simple to permit an experienced engineer to optimize system performance by adjusting parameters in the model by trial and error.

However, the advent of computer-controlled systems has opened up a new and very complex frontier of systems with many inputs and many outputs. The prototypical example of enormous complexity for such systems is the design of very large scale integrated circuits. The problem of finding efficient control laws for such systems has been approached by means of the Kalman digital filter and, more recently, by an extension of interpolation theory to matrix-valued analytical functions and to a full calculus of operators.

One benefit of the new theories has been to increase the flexibility of control system design. Instead of being restricted to traditional methods introduced by Norbert Weiner that minimized mean square error, designers can choose, with these new systems, to minimize worst case errors. Such options are of crucial importance when prevention of failure (for example, in nuclear power plants or in aircraft control) is of primary importance. Such benefits will come as circuit design shifts from scalar to matrix patterns, based on highly sophisticated mathematical theories of interpolation.

The availability of computers to carry out complex calculations has opened up many additional frontiers in control theory. For example, the theoretical basis for reconstructing an optimal signal from distorted versions received by multiple detectors uses techniques from the theory of analytic functions in several complex variables. New projective (Karmarkar-like) algorithms for linear programming have made possible automated control of rapid processes (for example, high-performance aircraft) not previously possible. Certain nonlinear control problems, such as the motion of an airplane while engaged in short landing and takeoff, can often be transformed via Lie algebras into a linear feedback law that can then be computed with the use of traditional methods that are both computationally stable and sufficiently simple to carry out in real time.

Stochastic differential equations. The laws of nature are expressed most eloquently in the language of differential equations. Maxwell's equations for electromagnetic fields, Newton's equations for planetary motion, and Navier-Stokes equations for fluid flow are as articles in nature's constitution. They express the way nature changes in terms that make possible both mathematical analysis and scientific investigation.

But in practice, the data available to a scientist are never known exactly and may be subject to significant random variations. In some cases, key variables are totally unknown or are obscured by noise. A common case, represented mathematically by what are known as stochastic differential equations, occurs when the system is subject to external white noise. Typical examples include diffusion processes in communication theory, chemical processes, stock market analyses, epidemiology, queueing theory, and population genetics.

All these examples share certain common features. Most important, like a fair coin whose probability of heads does not depend on the outcome of previous tosses, they are processes without a memory of the past. In mathematical jargon, they are what are known as Markov processes. In addition, they are subject to random influence of key variables—whence the stochastic nature of the problem.

Stochastic differential equations are one of the key areas of probability theory. Recent research has yielded important links to other parts of mathematics. In particular, certain statistics of the stochastic systems (for example, mean exit times of a diffusion process) turn out to be solutions of ordinary (nonstochastic) partial differential equations. This in turn has yielded interesting insights linking random processes in one system with geometry and analysis in another, notably mathematical physics.

The behavior of the solar system under the random influence of passing comets and stars is an example of a dynamical system governed by a stochastic differential equation. Since random disturbances may destabilize dynamical systems (even if the disturbances are small), the investigation of stability is a matter of utmost importance. Only recently have researchers been able to determine conditions under which the trajectories of solutions to stochastic flows will cluster in stable patterns. [Some researchers think the stock market is subject to the same types of instability, in which behavior near a "strange attractor" can lead to sharp oscillations that are inherently unpredictable (*24*).]

Physics. Reconciling quantum field theory with general relativity—the physics of the small with the physics of the large—is perhaps the major theoretical problem in physics. Because of the enormous difference in scale between gravitational effects and quantum effects, realistic experiments are of little value in suggesting avenues for exploration.

The leading current attempt at reconciliation of microcosmic with macrocosmic physics is the theory of strings, a construct in which the dimensionless points of four-dimensional space-time are replaced by thin strings in ten dimensions, where the intrinsic structure of the extra dimensions, like that of what we perceive to be "empty" space, occurs at a scale too small for our perception. Pursuit of this hypothesis (which can probably never be tested by experiment) leads naturally to higher dimensional symmetries in which are embedded the special symmetries, the fundamental invariants, of both macro- and microscopic physics (*25*).

Knot theory. Fifty years ago mathematicians developed the theory of operators, motivated in part by the need to find mathematical models of quantum mechanics. Operator theory flourished as a branch of functional analysis, which was pursued both for its pure mathematical interest and for its continued applicability to quantum physics. In recent years, investigation of new types of operators yielded new methods of classification that in turn have led to the discovery of important relations between operator algebras and the classification of knots, a vexingly difficult problem that had previously defied all attempts at solution.

The key to classification of knots is a scheme to encode knot patterns in algebraic terms so that algebraic manipulations correspond to physical actions on the knot (*26, 27*). This makes it possible, for the first time, to determine whether one knot can be transformed into another or unlinked completely into a straight line. Underlying both the new types of operators and the new classification of knots is a new algebraic structure whose characteristics were revealed by its appearance in knot groups, in certain areas of statistical mechanics, and in exactly solvable models. As happens so often in mathematics, the significance of the new structure emerged in recognition of its ubiquity: that the same pattern appeared in several places is precisely why the technique takes on special power.

Recently biologists studying the replication of DNA have teamed up with mathematicians working in knot theory because DNA in the cell is normally coiled into a tight knot. How DNA can replicate and then pull apart if it is tightly knotted is difficult to imagine—like the magician's trick of effortlessly separating two intertwined rope knots. From motivation and application in quantum mechanics through esoteric research in pure mathematics and then to the unfolding of DNA is an amazing, albeit not atypical, example of the many interconnections among diverse parts of mathematics.

Patterns

These examples from contemporary mathematical science illustrate metaphors of the mathematical method that originated 300 years ago in the Newtonian synthesis: data, deduction, and observation. They also reveal the effects of the major forces for change in the mathematical sciences: computers, applications, and cross-fertilization. Hundreds of other examples could have illustrated the same points; those chosen here are neither the deepest nor the most important. They do suggest, however, the variety and scope of today's mathematics.

Mathematics is often defined as the science of space and number, as the discipline rooted in geometry and arithmetic. Although the diversity of modern mathematics has always exceeded this definition, it was not until the recent resonance of computers and mathematics that a more apt definition became fully evident.

Mathematics is the science of patterns. The mathematician seeks patterns in number, in space, in science, in computers, and in imagination. Mathematical theories explain the relations among patterns; functions and maps, operators and morphisms bind one type of pattern to another to yield lasting mathematical structures. Applications of mathematics use these patterns to "explain" and predict natural phenomena that fit the patterns. Patterns suggest other patterns, often yielding patterns of patterns. In this way mathematics follows its own logic, beginning with patterns from science and completing the portrait by adding all patterns that derive from the initial ones.

To the extent that mathematics is the science of patterns, computers change not so much the nature of the discipline as its scale: computers are to mathematics what telescopes and microscopes are to science. They have increased by a millionfold the portfolio of patterns investigated by mathematical scientists. As this portfolio grows, so do the applications of mathematics and the cross-linkages among subdisciplines of mathematics.

Because of computers, we see more than ever before that mathematical discovery is like scientific discovery. It begins with the search for pattern in data—perhaps in numbers, but often in geometric or algebraic structures. Generalization leads to abstraction, to patterns in the mind. Theories emerge as patterns of patterns, and significance is measured by the degree to which patterns in one area link to patterns in other areas. Subtle patterns with the greatest explanatory power become the deepest results, forming the foundation for entire subdisciplines.

Texas physicist Steven Weinberg, echoing Harvard mathematician Andrew Gleason, suggests that the reason why mathematics has the uncanny ability to provide just the right patterns for scientific investigation may be because the patterns investigated by mathematicians are *all* the patterns there are (*23*). If patterns are what mathematics is all about, then the "unreasonable effectiveness" of mathematics may not be so unreasonable after all.

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