numbers. But, says Mazur, "the more general definition of a unit comes into play when mathematicians look at what they call algebraic integers. These are expressions that act like integers but are a bit more complicated."

An example of a system of algebraic integers is the collection of all numbers of the form $A+B\sqrt{2}$, where A and B are ordinary integers. You can add and multiply numbers of this form and the answer will always be a number of the same form. The units of a system of algebraic integers are those algebraic integers whose reciprocals are also algebraic integers. So, for example, in the system of algebraic integers of the form $A+B\sqrt{2}$, $1 + \sqrt{2}$ is a unit since its reciprocal is $-1 + \sqrt{2}$. In fact, Mazur notes, any unit in this system is either a power of $1 + \sqrt{2}$ or a power of its reciprocal or the negative of any of these.

Other systems of algebraic integers can be more complicated and their units can be difficult to determine. Harold Stark of the Massachusetts Institute of Technology and the University of California in San Diego conjectured that there is an amazing relationship between certain expressions involving the logarithm of these units of algebraic integers and the behavior of particular functions, called non-Abelian L functions, L(s), at the point where s is 0. He proposed that you can start with these functions from analysis, L(s), and get expressions involving the logarithms of units of algebraic integers and, in special cases, get the logarithms of the units themselves. What is surprising is that Stark thereby connects analysis and number theory.

Stark showed that his conjecture is true in some cases. And although no one yet has an inkling of how to prove the conjecture in general, work on it has suggested several new mathematical ideas.

Tate recently wrote a book on these developments and proved the conjecture in another special case. Tate's proof then led to new questions about the units of algebraic integers and Chinburg reported on recent research on these questions.

Other talks at the meeting were on problems relating number theory to algebra, algebraic geometry, and analysis. The theme, says Mazur, "is connecting what once seemed to be the unconnectable"—a possiblity that is bound to be exciting.—GINA KOLATA

When Are Viscous Fingers Stable?

Recent research concludes that a single, stable finger can form when a lower viscosity fluid pushes against one of higher viscosity

Interest in viscous fingering, a decades-old problem in the fluid dynamics literature, has taken on new life in the last few months. Activity is on two fronts, which already look as though they are quite closely related.

In the first area, theorists find they can now explain the persistence over long times of the distinctive finger patterns, although a complete quantitative description of the shapes is not yet in hand. At the same time, experimentalists and theorists have been jointly exploring the limits of the fingers' stability and find it to be not unlimited. A low surface tension at the interface between the two fluids, a large interface velocity, and fluctuations or noise at the interface all degrade the stability and give rise to distortions in the fingers in both numerical simulations and experiments.

The second area concerns the recently fashionable topic of fractal behavior in physical systems. Fractal objects have a property called self-similarity—that is, they have similar features at all length scales and therefore look the same at all magnifications—and are characterized by an effective fractional dimension, rather than the integer 1, 2, and 3 of curves, surfaces, and volumes.

Self-similarity is a kind of symmetry, in this case invariance under a change in length scale. This sort of symmetry was a crucial ingredient in the development of the theory of phase transitions (critical phenomena) over a decade ago, and more recently it has figured in the behavior known as chaos in nonlinear dynamical systems (*Science*, 5 November 1982, p. 554). Now scientists hope that it will play a similarly powerful role in understanding other physical phenomena. The recent observation of fractal viscous fingers, to be discussed in a second article, is therefore causing much excitement.

The question of the stability and shape of viscous fingers is part of a more general domain of inquiry called pattern formation. Physical systems that evolve under conditions far from equilibrium often take on characteristic shapes or patterns that are governed by a balance between competing forces acting on the system during its growth. Fluids are an especially fertile ground for such processes, as exemplified by the wonderful patterns of whorls and swirls that occur at flow rates between the laminar and completely turbulent regimes (*Science*, 8 July 1983, p. 140).

The mathematically simplest example and hence the prototype whose understanding should help with the solution of more complex problems in pattern formation is the single finger that can occur when a lower viscosity fluid pushes against one of higher viscosity.

A key event in the modern history of viscous fingering was a 1958 publication by Philip Saffman (now at the California Institute of Technology) and the late Sir Geoffrey Taylor of the University of Cambridge describing the displacement of a viscous fluid, such as oil, by a less viscous fluid, such as water, in a cell comprising two closely spaced flat plates. This flat-plate configuration is called a Hele-Shaw cell, after the British engineer J. H. S. Hele-Shaw, who invented it in 1898. As a kind of two-dimensional wind tunnel for liquids, it was useful for studying fluid flow past obstructions, such as ship hulls.

Saffman and Taylor were more intrigued, however, that the mathematics of their two-fluid experiment was also the same as that for flow in porous media, a problem of great interest to petroleum engineers and civil engineers. For example, one of the methods of enhancing the productivity of an oil field is by pumping water or carbon dioxide gas into the ground through one well in order to force more oil to flow to neighboring wells. The formation of fingers of water or gas in the oil, as was observed to occur in laboratory models, plainly affects the efficiency of the recovery, although there are probably other equally important factors in real oil fields, such as faults in the rock.

Whether or not the Hele-Shaw cell really does provide a model system for fluid flow in porous media, the distinctive fingering patterns investigated by Saffman and Taylor have emerged as a subject worthy of investigation in their own right.

In the case where the viscosity of one fluid is much less than that of the other, Saffman and Taylor found that an initially flat interface between the two became unstable as the less viscous fluid pushed against its counterpart; that is, small bumps formed and grew into fingers. However, one of the fingers quickly became dominant, so that after some time the others withered, leaving the victor to grow at a steady velocity dictated by the pressure on the fluid.

Upon reaching this steady state, which occurred for a wide range of velocities, the finger occupied about half the width of the Hele-Shaw cell as it continued its march down the length of the channel. A typical experiment took a half-hour or so for a cell 91 centimeters long, 2.54 centimeters wide, and 0.8 millimeters thick.

On the theoretical side, Saffman and Taylor found a continuous family of analytical solutions for the steady-state finger shape in the limit of zero surface tension at the interface between the two fluids. Although there was no unique solution for the finger under a given set of conditions, the one whose width was half the cell width matched the experimental profile quite well.

While the absence of surface tension makes the mathematics tractable, surface tension is one of two competing forces whose balance determines the finger size and must ultimately be taken account of.

In brief, the velocity at any point in either fluid is proportional to the pressure gradient there. Because the gradient is highest just ahead of an advancing finger, growth of the finger is further enhanced. This is why a bump on the initially flat interface grows into a finger in the first place. However, as the finger elongates, the interface area accordingly also increases, whereas the surface tension attempts to minimize the area. Surface tension therefore inhibits growth of multiple fingers and acts as a stabilizing influence on a single finger.

The following year, Taylor and Saffman pointed out a further complication. By means of a technique called linear stability analysis, in which it is possible to determine if small perturbations will grow and hence destabilize a pattern, the investigators found that all the mathematical finger solutions were unstable. This finding, which means that the finger patterns should not reach a steady-state condition but should break up into something more complex and less orderly, was in complete disagreement with the experimental results.

The next chapter in the question of the stability of viscous fingers came in 1981, when John McLean (now at TRW, Inc., Redondo Beach, California) and Saffman carried out numerical calculations at Caltech that included surface tension. In their calculations, the investigators obtained the steady-state shapes of viscous fingers, finding a continuous increase in the finger width as the surface tension increased. But, once again, linear stability analysis turned up the conclusion that these numerical finger solutions were unstable to perturbations.

Surface tension is one of two competing forces whose balance determines the finger size.

One explanation for the discrepancy between theory and experiment could be that for some reason the steady-state solutions arrived at theoretically are not the same ones the physical system of two viscous fluids actually chooses. Subsequent analysis by J. M. Vanden-Broeck of the University of Wisconsin suggested, for example, that there is a multiplicity of solutions for the steady-state equations. One way of avoiding unnecessary assumptions is to calculate the actual time behavior. The main difficulty is that the equations of motion together with the boundary conditions comprise a nonlinear problem that is not soluble analytically and requires a clever technique and a big computer to simulate.

Two years ago, Grétar Tryggvason (now at the Courant Institute of Mathematical Sciences in New York) and Hassan Aref of Brown University applied a specific version of a more general numerical technique called the boundary integral method to the viscous finger problem (*1*). Boundary integral method refers to a property of many so-called stratified flow systems that it is only necessary to track with time the evolution of the interface between the fluids. The fluid flow away from the interface is uniquely determined by the interface motion. This feature greatly reduces the computational burden. In the context of viscous fingering, the specific approach adopted by Tryggvason and Aref, following the earlier work of G. de Josselin de Jong at the University of California at Berkeley, takes the name vortex sheet method because the only place the fluids exhibit any vorticity is at the interface.

The approach was to generate small perturbations in a flat interface and calculate their subsequent time evolution. The perturbations were in the form of randomly selected collections of sine waves of varying wavelength. Their simulations included fluids with identical or nearly identical viscosities, as well as those with a large viscosity contrast. In the limit where the viscosity of one fluid was vanishly small, Tryggvason and Aref found the same type of single finger observed experimentally (although these and subsequent simulations generated narrower fingers than the steady-state calculations) and no evidence for instabilities.

The single finger that dominates grows from the so-called most unstable perturbation, which has a wavelength fixed by the surface tension and the velocity of the interface, the viscosities of the fluids, and the width and thickness of the cell. Fluid dynamics, in fact, is notorious for its dimensionless numbers, and these quantities make up such a number for the Hele-Shaw cell. Dimensionless numbers determine scaling properties of the fluid flow. In this case, for example, the number is called the capillary number and is proportional to the ratio of the interface velocity and the surface tension, which means that reducing the surface tension by a factor of 2 or doubling the velocity have identical effects. Since experimentally it is hard to change the surface tension, one achieves the same end by applying different pressures to the fluid in the cell and thereby changing the velocity.

The results of Tryggvason and Aref would seem to imply that the viscous fingers are in fact stable and that the earlier linear stability analyses were somehow in error. This year at least three groups have looked more deeply at the finger stability problem and the resolution may finally be at hand.

First, David Kessler of Rutgers University, Piscataway, New Jersey and Herbert Levine of Schlumberger-Doll Research, Ridgefield, Connecticut returned to the steady-state finger solutions and reformulated the stability analysis. They report that, when the surface tension is properly accounted for, the solutions are linearly stable after all, provided that the surface tension is not so small as to invalidate the numerical methods used (*2*).

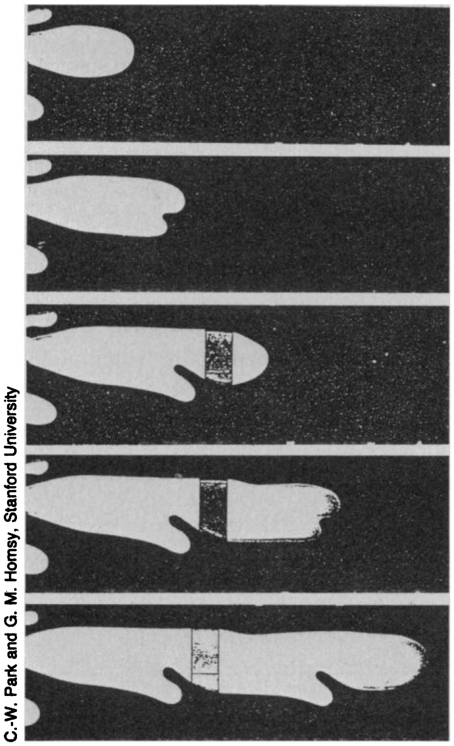
Apparently, the reason that the earlier linear stability analyses reached the incorrect conclusion is that the surface tension was introduced only as a small parameter, a kind of perturbation itself, to use in an expansion around the zero–surface tension solution. Kessler and Levine argued that the introduction of any finite surface tension completely alters the analysis. Inclusion of the surface tension made it impossible to do the stability analysis analytically, but numerical solutions were obtainable.

Second, at the Exxon Research and Engineering Company, Annandale, New Jersey, Anthony DeGregoria and Leonard Schwartz attacked the stability issue by numerically simulating the growth of small perturbations on the tips of fingers having the Saffman-Taylor steady-state shapes. They concluded that the fingers were linearly stable (that is, small perturbations did not grow) but were nonlinearly unstable (large perturbations did grow) (*3*).

The Exxon calculations proceeded in two parts. The first was to reproduce the fingers in a way similar to that of Tryggvason and Aref; that is by following the time evolution of a perturbation on a flat interface, in this case a single sine wave. However, DeGregoria and Schwartz used a different numerical implementation of the boundary integral method that allowed them to simulate very low surface tensions. When the surface tension is low, the simulations require high spatial resolution, which means that more points have to be calculated to accurately trace the interface, and this increases the computational burden. Second, to determine the stability of the fingers, the theorists introduced deformations, such as notches, near the tip of the finger and followed their fates.

Small notches placed at the tip were found to initially grow, but they subsequently decayed. Presumably, the traditional linear stability analysis would have concluded that these perturbations would destabilize the fingers because the analysis would only "see" the early growth stage. According to the simulations of DeGregoria and Schwartz, the perturbations initially grow because of the high pressure gradient at the finger tip. However, as the tip continues to move down the Hele-Shaw cell, the perturbation slides down the side of the finger to a region of lower pressure gradient, where the relative influence of the surface tension is larger and its growth is dampened.

Larger notches caused a phenomenon known as tip splitting, in which the finger bifurcated and then continued to grow as a symmetric pair of fingers. It was also possible to introduce asymmetric bifurcations in which only one of the pair continued down the channel, while the other remained in a vestigial form. In either case, the proposed explanation is that large perturbations initially can grow enough to dominate the subsequent time evolution of the finger before sliding into the damping regions down the sides of the finger.



C.-W. Park and G. M. Homsy, Stanford University

Tip splitting

Sequence of growth stages for viscous fingering when the surface tension at the interface between the fluids is low. Times are 90, 110, 140, 171, and 210 seconds after the start of the experiment.

Third, at the University of Chicago, David Bensimon has calculated in a linearized model the effect of noise on the growth of viscous fingers (*4*). Experimentally, one does not neatly cut notches into the tip of a liquid finger, but imperfections in the experimental apparatus effectively do this automatically. It is extremely difficult to build a Hele-Shaw cell with perfectly flat and parallel plates, for example. Researchers call the random imperfections introduced in this way noise. In numerical simulation, noise is introduced by making the interface position vary randomly.

Bensimon found that the noise needed to destabilize a growing finger decreases exponentially as the interface velocity increases or, equivalently, as the surface tension decreases. In this way, they have come full circle to the original analysis of Saffman and Taylor. As the surface tension vanishes, no noise at all is required to destabilize the fingers. Numerical simulations at Chicago similar in spirit to those at Exxon, where noise has also been seen to have a destabilizing effect on otherwise stable fingers, confirm these conclusions.

Experimental confirmation of these instabilities is a tricky business, partly because of the huge effect that the details of the wetting of the walls and plates of the Hele-Shaw cell can have on the formation and evolution of the fingers. All the theorists used a quite simplified model of this wetting. Nonetheless, confirmations are beginning to come in. At Stanford University, for example, Chang-Won Park (now at Union Carbide, Bound Brook, New Jersey) and George Homsy have recently completed experiments using a much wider Hele-Shaw cell than Saffman and Taylor did in the original experiments (*5*). According to the scaling in the dimensionless capillary number relevant to flow in the Hele-Shaw cell, increasing the cell width has the same effect as decreasing the surface tension, thereby making observation of the tip-splitting instability easier.

For small capillary number (large surface tension), the Stanford experimenters found fingers of air pushing into a glycerine-water mixture of the same size and shape as seen by Saffman and Taylor. As they increased the capillary number, however, they observed the onset of an asymmetric tip splitting in which the dominant subfinger grew to the size and at the rate of standard, stable fingers (see photo). Tip splitting occurred at regular intervals, so that this scenario repeated itself periodically.

In experiments still in progress at Chicago that are aimed at quantitatively checking the various theoretical predictions, Patrick Tabeling and Albert Libchaber are seeing quite similar effects. Above a certain value of the capillary number, they observe an asymmetrical deformation in which one side of the finger grows a little more than the other. Tip splitting occurs at still higher capillary numbers.

Taken together, these investigations seem to be putting a lid on one aspect of the viscous fingering. The recent work suggests that stable fingers exist over a wide range of capillary numbers, including very large values, but that they become increasingly liable to small perturbations as the capillary number grows. If a stable finger does not form, the fluid behavior becomes much more complex and may exhibit fractal behavior, as will be discussed in the second article.

—ARTHUR L. ROBINSON

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