overlying the gently inclined subducting ocean crust. A section of ocean crust apparently broke off and underlies the accreted terranes. Prominent reflectors in recently acquired data coincide with the upper-lower crust boundary, the bottom of the detached oceanic plate, and either the continental Moho or the top of the subducting ocean plate.

While the USGS and non-U.S. programs emphasize the use of multiple techniques at the same sites, COCORP speakers placed a new emphasis on the reconnaissance nature of most of their program. Refraction will help, but the ultimate test of seismic methods—deep drilling—is only now being proposed. A 10-kilometer hole in the southern Appalachians (*Science*, 29 June, p. 1418) would help determine what some reflectors really are.—**RICHARD A. KERR**

Esoteric Math Has Practical Result

A new method of computer graphics relies on math results that seemed so abstruse that they were never published

The problem with natural objects is that they are so irregular. When programmers try to tell a computer how to draw a cloud or a leaf or a forest, they run into difficulties. If they attempt to specify each and every detail, they will come up against a monumental computing task. It can take thousands or millions of bits of stored data to draw a realistic scene and computers quickly run out of space. It also takes computers a long time—18 hours in some cases—to put all this stored data together to make a picture. If programmers try to provide general rules for drawing scenery, the computer pictures will look a little too smooth and regular. It is even more difficult to solve what computer scientists call the encoding problem. Take a scene, digitize it, and compress the information substantially so it can be easily stored. Then ask the computer to recreate that exact scene any time you want to see it.

But Mehrdad Shahshahani, a mathematician at Boeing Aerospace in Seattle has an extremely promising approach to solving both of these problems. He has found a way to make computer pictures of natural objects and to encode pictures of scenery with very little effort. He can generate a realistic picture of a leaf, for example, with only 21 numbers and three simple equations. Boeing Aerospace wants to use Shahshahani's results in its flight simulators, which are computerized systems used to train pilots by giving them the exact sensations of flying, complete to the scenery outside the window, without ever leaving the ground.

Shahshahani's work relies on some very esoteric abstract mathematics, which seems so unrelated to the real world that when Persi Diaconis, a statistician at Stanford University, studied this math 10 years ago, he decided not even to publish his results. But the mathematics results turned out to be just what is needed to determine which numbers and simple equations will make which pictures of natural objects. The story of how this mathematics came to light is the sort of story that is dearest to mathematicians' hearts. It is a story of mathematics pursued for its own ends that eventually finds an unexpected and significant use.

In 1974, as a graduate student in the statistics department of Harvard University, Diaconis at first had difficulty finding a research problem that interested him. Then, by chance, he came upon the "first digit problem," a problem first described around the turn of the century by Simon Newcomb, an astronomer. Newcomb was led to a curious result about the distribution of the first digits of numbers when he noticed that the beginning pages of books of logarithms were the most worn, indicating that people were looking up more logarithms of numbers starting with 1 than any other number.

If you look at the lead digit in any source of numbers, such as the pages of *Science* or the numbers in the almanac, you might expect that the number 1 would turn up about one-ninth of the time. After all, there are nine possible first digits and there is no reason to believe that any one digit would be favored over any other. But, surprisingly, the number 1 is the first digit about three times out of ten because the numbers that begin with 1 are irregularly spread among all the numbers. So, for example, one-ninth of the numbers from 1 to 9 begin with 1. One-half of the numbers from 1 to 20 begin with 1. One-ninth of the numbers from 1 to 100 start with 1. One-half of the numbers from 1 to 200 start with 1. As you look at larger and larger sets of numbers, the proportion of numbers in the sets with lead digits that are 1 oscillates between one-half and one-ninth. Diaconis asked whether there was some other natural way to take an average so that the average number of lead digits that are 1 will settle down rather than oscillate.

A way to do this, Diaconis found, is to use the Riemann zeta function, which has been the object of intense study for the past century because if more were known about it, more would be known about where prime numbers lie. The zeta function is an infinite sum, and Diaconis found that if he used the terms of that sum as weighting factors, he could get a way of averaging that would avoid the oscillations in the first digit problem. At the same time, this averaging method would give the usual sort of average in cases where the average does not oscillate. For example, both it and the ordinary way of taking an average say that one-half of all whole numbers are even. His method of "zeta averages" says that the density of the set of numbers that begin with 1—the chance that if you pick a number at random it will begin with 1—is $\log_{10}2$, or .301.

"This was very esoteric math," Diaconis says. "It was the sort of math that made people say, 'Gee, that's funny, but why would anyone care?' " In fact, when Diaconis went to the University of California at Berkeley in 1973 to give a talk on his thesis as part of a job interview, he recalls the Berkeley statisticians saying, "We assume you'll find something else to work on."

Diaconis accepted a job at Stanford rather than at Berkeley and he did find many other things to work on. He all but forgot the first digit problem. In the meantime, Shahshahani was investigating a highly innovative way of producing computer graphics. He got his inspiration from some work done 3 years ago by John Hutchinson of the Australian National University in Canberra. Hutchinson was interested in generating fractals, which are mathematical entities with fractional dimensions. "There was no indication that he wanted to use his method for pictures," Shahshahani says. But Shahshahani recognized that Hutchinson's ideas could be used for computer graphics.

What Shahshahani does is to take very ordinary curves and lines and repeatedly apply certain simple transformations, called affine transformations, which deform them. Then he looks at the fixed points, which are those that do not move under the transformations. By plotting the fixed points he generates pictures of natural objects.

Shahshahani came to Diaconis to gain more insight into why the procedure works and how to best choose the original transformations. Diaconis looked at the problem. "I sat there and shook my head. The way to solve the problem is to use exactly the crazy results from my thesis that I never published."

Shahshahani's method is to start with a few particular affine transformations, which are operations that take a line in the plane and first contract and rotate it and then shift it to a new position. He applies the first transformation to a line and plots the fixed point. Then he applies the second transformation and plots the fixed point. Next he multiplies the first transformation times the second, applies the resulting transformation to the line, and plots the fixed point. He then multiplies the second transformation times the first (multiplication of transformations is not commutative so the product of the first and the second is different from the product of the second and the first) and repeats the process of plotting the fixed point.

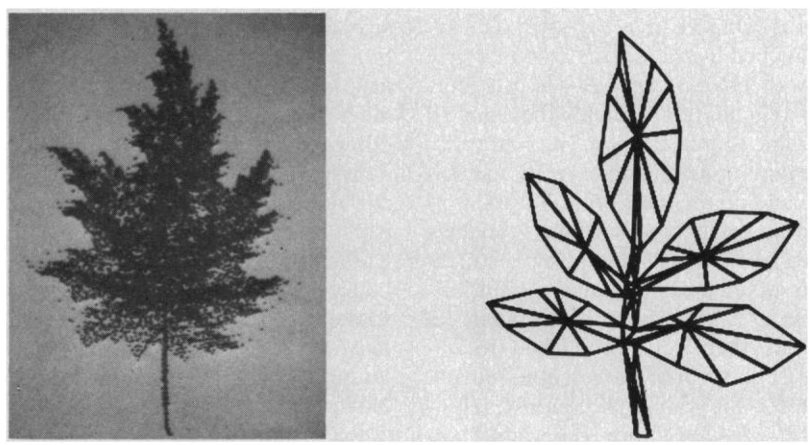
The next step is to multiply every combination of the original transformations when they are grouped in threes, such as $1 \times 2 \times 1$, $2 \times 1 \times 1$, $1 \times 1 \times 2$, and so on. Once again, these resulting product transformations are applied to the line and the fixed points are plotted. Shahshahani stops the process when he gets enough points for a realistic picture.

When Diaconis looked at Shahshahani's procedure, he recognized a math problem that can be expressed in the language of Markov chains—sequences of events with the property that each event in the chain depends only on the preceding one. They are widely used in statistics to model random phenomena.

In Shahshahani's case, the Markov chain was the sequence of affine transformations. The process of picking the transformations in the sequence gives the same result whether it is done deterministically or randomly. For example, you could flip a coin to decide whether to start with transformation 1 or 2. Then when you are multiplying groups of two transformations together, you could flip a coin to decide what order to multiply them in.

The fundamental theorem in Markov chain theory says that eventually the chain has to settle down into a stationary distribution. For example, Diaconis says, if he is mixing the cards in a deck and he starts with the cards in a known order, he may proceed by switching two cards at random. Then he may switch two other cards. As he continues in this way, the arrangement of cards in the deck will get more and more random. In that case, the stationary distribution is a random mixture of cards.

To find what initial transformations will lead to various stationary distributions required analyzing random infinite sums—the sums being the transformations. "It was precisely like the sums in my thesis," Diaconis remarks. "I called Mehrdad and said, 'I know all about these things!' " Based on Diaconis' results from analyzing the first digit problem, he and Shahshahani can predict stationary distributions and so are able to tell what computer pictures will look like on the basis of the original transformations. More importantly, they can solve the harder "inversion" problem of telling what transformations will give rise to particular pictures of natural objects.



The new and the old

The leaf at left was drawn with the new "fixed point" method. It required 12 numbers for the stem and 21 for the main part of the leaf. The leaf at right was drawn in the conventional way, by specifying the lines for the computer to draw. [*Sources: Merhdad Shahshahani (left); Cranston/Csuri Productions, Columbus, Ohio (right)*]

Once Diaconis expressed Shahshahani's problem in the language of Markov chains, he saw that what Shahshahani really wanted to know was the stationary distributions he will get with various initial transformations. That would tell him how his choice of affine transformations affects the computer picture.

"The quality that Mehrdad's pictures have is that they are 'leafy.' They don't smear out over the screen. They have a delicate fine structure no matter how closely you look," Diaconis remarks. "The main issue was to determine the mathematical notion that captures this property." Diaconis believes that the key to leafyness is a notion of being "singular continuous," a mathematical phenomenon, Diaconis says, "that was always thought to have no application whatsoever." Shahshahani disagrees, arguing that it is other aspects of the distribution that are important. Both ideas, however, can be investigated with the research the two are undertaking.

Shahshahani is developing computer programs to implement the inversion procedure. Now, he says, that "the math is under control," he expects within a few months to be able to make pictures of simple natural objects, digitize them, and reproduce them exactly on the basis of very little stored information. This will be "not just a leaf or a tree that looks nice; you tell me what kind of leaf you want and I should be able to generate it."

Diaconis, in the meantime, is rereading some papers on random infinite sums published in the 1930's by the Hungarian mathematician Paul Erdös. "Erdös studied them for no reason other than that they were beautiful and interesting. At the moment, I'm going over Erdös's papers with a fine tooth comb," Diaconis remarks. It looks as if the more that is known about these seemingly esoteric quantities, the more will be known about how to encode and generate computer pictures of natural objects.

—GINA KOLATA