Graph Theory Result Proved

Two mathematicians, working on a counterintuitive conjecture resolved a problem that has been around for 50 years

About 20 years ago, the German mathematician Kurt Wagner made a conjecture about the properties of graphs that sounded so incredible that even his graduate students did not believe it. But it was intriguing and of such fundamental importance that many mathematicians set out to determine whether he was right or wrong. Among these investigators was Neil Robertson of Ohio State University who says he has been intrigued by Wagner's conjecture for 20 years, his entire career as a researcher. But now Robertson and his colleague Paul Seymour, who has recently moved to Bell Communications Research in Murray Hill, New Jersey, have the first glimmer of success. They have proved that Wagner is right in many special cases and, in doing so, have proved another open conjecture that has stumped mathematicians since the 1930's. Their result, says Ronald Graham, director of mathematical sciences research at AT&T Bell Laboratories, is "quite strong."

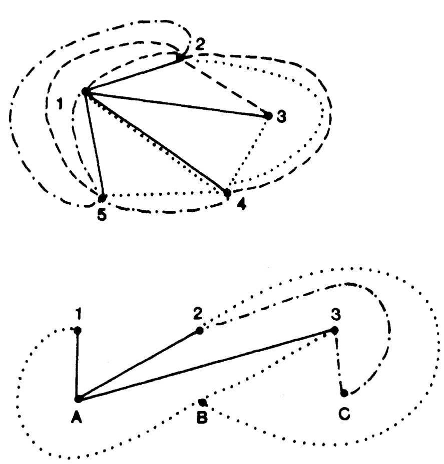
The story begins in 1930 when the Polish mathematician Kasimir Kuratowski asked what is it that makes certain graphs—collections of points connected by lines—impossible to draw on the plane without any of the lines crossing. Mathematicians already knew that there are many graphs that cannot be drawn on the plane, but no one knew what it was that gave the graphs that property.

This is not just a theoretical problem. For example, circuits on computer chips are essentially graphs, with the transistors as the "points" and the conductors as the "lines." Today's designers of very large scale integrated circuits need to know if there is some way of laying down channels of conductors between transistors so that none of the channels cross.

In Kuratowski's day, mathematicians had come upon some simple examples of graphs that cannot be drawn on the plane, including two graphs that seemed to be the simplest of all. One of these consists of just three points, labeled A, B, and C, and another set of three points, 1, 2, and 3. It is impossible to connect each lettered point to each numbered point without having some of the connecting lines cross—unless, that is, you have some lines loop up in the air out of the plane. Another example is with five points. It is impossible to connect each of five points to every other without having the lines cross or without leaving the plane. Kuratowski showed that it is actually these two graphs, which mathematicians now refer to as Kuratowski graphs, that make matters so difficult. Any graph that contains one of these two cannot be drawn on the plane, and any graph that does not contain one of these two can.

Then, in the early 1930's, the Hungarian mathematician Paul Erdös asked if Kuratowski's result could be extended. What, he asked, are the minimal graphs—those that do not contain any other—that cannot be drawn on other surfaces? The Kuratowski graphs can be drawn on more complex surfaces that typically are twisted or have handles protruding from them. The lines that will not fit on a plane can go through a handle, for example. Erdös knew there may be other, more complex graphs, that will not fit on these surfaces. But, he wondered, are there an infinite number of them, just as there are an infinite number of primes?

The question was far more difficult than it sounded and it was not until 1980 that anyone made a dent in it. Then, Daniel Archdeacon, Henry Glover, Philip Huneke, and C. S. Wang of Ohio State University showed that for the Möbius strip, made by taking a strip, twisting it, and then joining the ends, there are 103 minimal graphs, many of them made by sticking the Kuratowski graphs together. The Kuratowski graphs themselves can be drawn on the Möbius strip. The Ohio State researchers next showed that there are more than 800 minimal graphs for the torus, which is essentially a sphere with one handle. For a sphere with two handles, Seymour estimates, there are at least 80,000 minimal graphs. "It looked like a dead-end sort of problem," Seymour remarks. It was not even clear that the list of minimal graphs is finite, that it has to stop somewhere.



The Kuratowski graphs

In the upper graph, it is impossible to connect point 5 to point 3 without crossing lines. In the lower graph, points 1 and C cannot be connected. These graphs are the best you can do. Any attempt to connect every point with every other point in the upper graph or to connect each of the lettered points with each of the numbered points in the lower graph will fail. There will always be at least one line that will not fit in without crossing one of the other lines.

But, about a year ago, Archdeacon and Huneke discovered that the list has to stop for graphs on one kind of surface—those, like the Möbius strip, that have a twist. Mathematicians call these surfaces "non-orientable" because it is impossible to orient yourself on them—to tell whether you are on the "inside" or the "outside." Very recently, and as a consequence of their work on Wagner's conjecture, Seymour and Robertson proved that the list of minimal graphs is finite for all surfaces, orientable and non-orientable.

Wagner's conjecture actually contains the problems posed by Erdös. In the 1960's, he proposed that, for any property of a graph, the list of minimal graphs is finite. These properties include that of being unable to draw a graph on the plane with no lines crossing. Other important special cases of Wagner's conjecture, says Robertson, are the property of being highly connected, meaning that any two points on the graph have many paths joining them that do not meet each other, and not being four-colorable, meaning that it is impossible to assign one of four colors to each point on the graph and have no line of the graph connect two points of the same color.

Another way of stating Wagner's conjecture, Seymour points out, is to say that any list of graphs with no one graph contained in another must be finite. The conjecture sounded counterintuitive. "No one believed it," Seymour says. But he and Robertson finally hit on a way to get at it. They used the observation that if they take a list and want to prove it is finite, they can take out one of the members of the list and then use the fact that that member is not contained in any other in the list to say something about the remaining members. "It tells you something about the other members and it puts a lot of structure on the list," Seymour says. He and Robertson used this technique to show that lists with one member missing are finite and so with the member added back they must still be finite. It is not at all obvious that this method should have worked so well, Seymour notes and, in fact, he remarks, "It seems silly that it should be so helpful."

But once they discovered their method of attack, Robertson and Seymour were able to prove two variants of Wagner's conjecture. First, they showed that Wagner's conjecture is correct for any list of graphs that go on surfaces with an upper bound to the number of handles on them. Then they showed that it is true for lists of graphs that contain at least one planar graph.

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This work, say Seymour and Robertson, took a couple of years and has already resulted in seven papers, each about 40 pages long. More papers are to come. Each paper, Graham remarks is "dense." Seymour and Robertson used only pen and paper to get their result. "I wouldn't even know how to use computers for this work," says Seymour.

Now, according to Graham, Wagner's surprising conjecture is "looking better." And although it still seems impossible to list all the minimal graphs that cannot be drawn on particular surfaces, the fact that the list is finite "is an encouraging sign. We may be able to characterize them or to describe them in other ways." It is too soon to apply these results to practical problems such as those that occur in the design of computer chips, but, Graham notes, the results, at the very least, lead to a better understanding of what it is about a network that may require many layers of circuitry, and they are of great interest to mathematicians who are trying to understand the properties of abstract surfaces and how to characterize graphs that can be drawn on them.—GINA KOLATA

A New Kind of Epidemiology

In February, a group of epidemiologists published the results of a prospective study of possible risk factors for cancer. The participants gave blood samples at the start of the study and were followed for 5 years. But unlike other prospective clinical trials that cost millions of dollars, this one cost only $6000. It is among the first of what promises to be a slew of a new kind of epidemiological study called "retrospective case control"—studies that are reusing data and material (usually blood) from other prospective clinical trials to answer questions that the original studies did not address.

In the study published in February, for example, Walter Willett of Harvard Medical School and his associates used blood from participants in the Hypertension Detection and Follow-up Program (HDFP), a study of 10,940 men and women that was conducted by the National Heart, Lung, and Blood Institute. All of the participants were carefully followed, their causes of death recorded, and their blood samples stored.

Ten years after the start of this study, Willett and his colleagues went back and selected the participants who got cancer, chose for each of them two controls who were matched for age, sex, smoking history, month of blood collection, blood pressure at the start of the study, randomization in the study to treatment or control group, reported use of antihypertension medication, and, for the women, number of children and menopausal status. Using these cases and controls, they could then ask whether the persons who developed cancer had lower levels of vitamin A, vitamin E, or carotenoids in their blood at a time before their cancer was diagnosed. (*Science*, 16 March, p. 1161). They saw no such relationship, but, says Willett, "We set a new record for low costs."

At about the time that the HDFP study was begun, the NHLBI also initiated two other large prospective studies of heart disease. The blood and serum samples from those studies are being used now to address epidemiological questions. B. Frank Polk of Johns Hopkins University Medical School and his associates are planning to use the blood from the Multiple Risk Factor Intervention Trial (MRFIT) to look for a relationship between vitamins in the blood and risk of cancer. Basil Rifkind, director of the Lipid Research Clinics at the NHLBI, says that stored serum from the Lipid Research Clinic studies is now being analyzed to see if apoproteins in the blood are better predictors of heart disease risk than cholesterol or lipoproteins and also to look at the cancer and vitamins hypothesis.

Other investigators are looking at stored blood from still other studies. For example, George Comstock of Johns Hopkins University Medical School has over 25,000 blood samples from residents of Washington County, Maryland, which he has been saving since the early 1970's for retrospective case control studies. The county has a cancer registry, so he knows which of the residents developed cancer. He is looking at the vitamin hypothesis and also is collaborating with Nancy Gutensohn of the Harvard School of Public Health to pool his samples with samples from several large Norwegian populations, the HDFP, and the Kaiser-Permanente population to see if persons who develop Hodgkin's disease had antibodies to Epstein-Barr virus in their blood before they were diagnosed.

The appeal of retrospective case control studies is that they are so terribly cost effective. As Rifkind explains, instead of taking 5000 or 10,000 people, measuring the vitamins in their blood, and then following them for 5 or 10 years to see which get cancer, he is able to take 136 cancer patients, match each with two controls, determine the vitamins in the 408 serum samples stored from before the patients got cancer, and get his results almost immediately and with very little expense. In addition, most of the samples remain untouched and so can be used to answer other questions, such as the question about apoproteins and heart disease.

But, Rifkind cautions, these studies do not always work. Some substances that investigators want to study break down—two of the apoproteins are a case in point. So, he says, "You can't just do one huge prospective study, store the blood and serum for 15 years, and then answer every question you want. Although, some of us have thought of that."

—GINA KOLATA