Book Reviews

Cognitive Development

The Development of Mathematical Thinking. HERBERT P. GINSBURG, Ed. Academic Press, New York, 1983. xvi, 390 pp., illus. $37.50.

How children learn mathematics may currently be the most lively research area within developmental psychology. As illustrated in this volume, ideas from diverse sources have contributed to the progress and controversy. Topics vary from how preschoolers learn to count to how college students learn calculus. Methodologies range from clinical interviews to computer simulations. Populations studied include not only middle-class children from Western countries but ghetto children encountering difficulty in learning mathematics, children labeled "learning-disabled," and schooled and unschooled children of Third World tailors, merchants, and farmers. In sum, the area is a hybrid, blending strains of cognitive and developmental psychology, computer science, mathematics education, linguistics, and anthropology.

In the past five years, at least five monographs and four edited volumes on children's mathematical reasoning have been published. Before reading *The Development of Mathematical Reasoning*, I was skeptical about what one more edited volume would add. The book, however, evidences four types of progress that render it a real advance over previous efforts: increasingly sophisticated use of error analysis, simultaneous modeling of representation and process, documentation of the ways in which environments influence mathematical thought, and exploration of the relations among mathematical attainments.

Analyzing error patterns on mathematics problems has long been recognized as a powerful means for obtaining insight into the thought processes that lead to errors. Van Lehn's chapter grapples with an especially complex problem in error analysis—that a single difficulty may be manifested in a variety of specific erroneous procedures. Consider the following problems and answers:

$$\begin{array}{r} 102 \\ -\ 39 \\ \hline 73 \end{array} \qquad \begin{array}{r} 702 \\ -108 \\ \hline 504 \end{array} \qquad \begin{array}{r} 10012 \\ -\ 214 \\ \hline 808 \end{array} \qquad \begin{array}{r} 8001 \\ -\ 43 \\ \hline 8068 \end{array}$$

All of the errors involve borrowing from zero, but no single rule generates all of them. In the leftmost three problems, the child decremented by one the first non-zero number to the left of the point at which she needed to borrow; in the rightmost problem she did not decrement this number. In the third problem from the left, the child avoided the need to borrow by arbitrarily restoring the second rightmost column to its original value of one, even though she earlier borrowed from it and on her work sheet reduced it to zero. Nowhere else did she do this.

Van Lehn generated an ingenious account of these and numerous other errors. He suggested that many children possess a core subtraction procedure that operates effectively in most situations but that fails when a child must borrow from zero because the child does not know how to decrement the zero. To escape from this impasse, the child backs up to the last point at which the difficulty could have been avoided and adopts one of a number of "repairs" to allow the algorithm to take some action. In the example $10012 - 214 = 808$, the child indicated that $0 - 1 = 1$ in the second rightmost column not because she believed that this answer was correct but because it avoided a seemingly worse problem later in the procedure (if she followed her usual algorithm here, she later would need to borrow from three zeroes with no non-zero number to their left). The repair theory with backup presents an appealing mixture of systematic and ad hoc procedures and seems to me an unusually plausible depiction of problem-solving.

Previous research on the development of mathematical skills has focused on children's problem-solving processes or on their representations of mathematical knowledge but rarely on both. Chapters by Resnick, by Riley, Greeno, and Heller, and by Van Lehn go beyond this dichotomy to examine how representations and processes interact. Resnick illustrates how children's representations of sets in terms of parts and wholes is critical to their invention of the "min" process, a widely used addition algorithm. Riley *et al.* examine children's representations of a wide range of arithmetic story problems and examine how the representations influence the problem-solving process. Van Lehn's model, while emphasizing processes, also includes "critics," which are representations of standards for plausible answers. Formulating models that specify both representations and processes is a challenging and important problem throughout cognitive psychology. The presence of three attempts to deal with it is a noteworthy feature of this book.

As Davis notes in his chapter, the focus of research on mathematics learning has shifted in recent years from an emphasis on how to teach concepts to a concern with how children invent them. This shift has enabled researchers to discover many intriguing learning strategies but has discouraged attention to environmental influences. The chapters by Allardice and Ginsburg and by Saxe and Posner are valuable exceptions to this tendency. Allardice and Ginsburg review research on the mathematical understanding of children labeled learning-disabled. They find no evidence that these children have neurological disturbances or that their thought processes are fundamentally disordered. Instead, they suggest that the children's difficulties arise from being presented tasks that are too far beyond their existing knowledge. Saxe and Posner examine environmental effects in a cross-cultural context. They document how differences in counting systems influence how people solve simple arithmetic problems and how parents' economic activities influence children's mathematical understanding. Especially interesting is a demonstration that children's understanding of specific mathematical ideas such as ratios may be related to how extensively parents use the ideas in their everyday activities.

Two chapters are notable for their models integrating performance across a variety of tasks. Fuson and Hall present a unified account of learning of the counting string and of how the words in it are used to establish cardinal values of sets, to measure continuous dimensions, and to order items within sets. Resnick's chapter describes how successively deeper understandings of the part-whole concept first help children to invent the "min" algorithm, then help them to understand numbers as simultaneously being points on the number line and components of other numbers, and eventually help them to understand that numbers can be partitioned in a variety of ways, thus allowing them to grasp conceptually the borrowing procedure in subtraction.

The Development of Mathematical

Thinking thus reflects several advances in an already exciting field. It is well worth reading.

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Statistical Mechanics

Long-Time Prediction in Dynamics. Papers from a workshop, Lakeway, Tex., March 1981. C. W. HORTON, JR., L. E. REICHL, and V. G. SZEBEHELY, Eds. Wiley-Interscience, New York, 1983. xviii, 496 pp., illus. $85. Nonequilibrium Problems in the Physical Sciences and Biology.

To what extent is the Laplace ideal of a deterministic, mechanistic universe operationally useful? Are there conceptual barriers to predicting the future course of a dynamical system from current information? Answers presented in this volume of proceedings are in four sections, reasoning from the general to the specific. The first section, designated Statistical Mechanics, avoids both thermodynamic and narrowly technical aspects associated with the term, concentrating rather on the underlying framework of an ensemble of dynamical trajectories. This subject is a natural vehicle for discussion of the sensitivity of a system configuration to uncertainty in its prior state or states and leads to the now familiar classification of ergodic, mixing, and K systems and so on. A theme running through this section is the relationship between predictability and reversibility. Lebowitz describes how the irreversible Boltzmann equation limit for a dilute hard sphere gas is lost as the permissible initial ensemble required for its validity contracts in time, Misra and Prigogine operate within a solution set in which future-directed correlations are omitted once and for all, and Grad focuses on the full Boltzmann equation hierarchy to indicate how dynamical systems organize themselves in defiance of initial information. Goldstein surveys conditions under which an achieved stationary measure takes on the classical Gibbs form, and Ford analyzes the concept of randomness from the viewpoint of algorithmic complexity.

In the second section, the development of effective stochastic behavior is examined in much greater detail by reference to explicit dynamical systems. Helleman, MacKay, and Greene do this in the context of iterated area-multiplying two-dimensional maps, the first two authors by a continual rescaling (renormalization) that preserves the form of the dynamics and the last by a Cayley representation of 2 × 2 matrices. It is the parametric dependence of such maps that is in question, and the Feigenbaum cascade of period doublings as fixed points become unstable is brought out very clearly, as is convergence to the traditional one-dimensional logistic map when the transformation is dissipative or area-contracting. A transition is then made to continuous-time Hamiltonian dynamics of systems that can be regarded as perturbations of integrable systems presented in angle-action form. Integrability persists, confined to (Kolmogorov-Arnol'd-Moser) phase-space tori, which can disappear as the perturbation amplitude increases. The complementary regions are the home of chaotic trajectories. Escande examines by renormalization a two-variable case (with two competing attractors) masquerading as a time-dependent one-variable Hamiltonian, eliciting a fractal pattern of disappearing tori. Salat and Tataronis show that, for a linear oscillator with quasi-periodic frequency, phase space is integrable either everywhere or nowhere. Lieberman and Tennyson proceed to more than two degrees of freedom, where the KAM tori cannot isolate chaotic regions, now represented by an "Arnol'd web" developing from the zero-perturbation resonance hyperplanes. They discuss the wandering of trajectories via Arnol'd diffusion and modulational diffusion (due to joining of chaotic regions), as well as the effect of noise.

The remainder of the volume is devoted to an array of specific physical problems in which long-time behavior is crucial, starting at the end of the second section with discussions of stability of satellite motion by Szebehely and Vicente and proceeding to a number of studies motivated by problems in plasma physics and accelerator design. Grebogi and Kaufman, and Dubin and Krommes, introduce the Littlejohn noncanonical variable formalism to handle the effect of resonant perturbations on charged particle motion, and Ott shows how stochastic rays can increase accessibility to plasma heating waves. Horton discusses plasma turbulence from a diagram renormalization viewpoint, and Molvig *et al.* develop a stochastic reference model in a Lagrangian flow representation to treat this problem. The genesis of coherent soliton motion is examined by Ichikawa *et al.* in the context of Alfvén waves and by Hyman *et al.* for energy propagation in an α-helix protein.

A final section focuses on beam-beam interaction, with Kheifets analyzing modulational diffusion via the Chirikov criterion for fusion of stochastic layers surrounding resonances and Tennyson treating the same phenomenon from the viewpoint of resonant streaming. Bountis *et al.* reduce cylindrical beam collisions to a two-dimensional map, precluding the possibility of Arnol'd diffusion. The section concludes with an application by Rechester *et al.* of path integral techniques to the Chirikov-Taylor standard map in the presence of noise.

In summary, these proceedings succeed very well in immersing the reader in an active and stimulating field. They are certainly sketchy and are more to be read than studied. But the reader who allows himself or herself to be swept along in the flow of erudition will pick up a good intuitive feeling, a number of valuable techniques, and a desire to deepen his or her understanding by consulting the copious references with which the book is adorned.

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Hummingbirds

The Hummingbirds of North America. PAUL A. JOHNSGARD. Smithsonian Institution Press, Washington, D.C., 1983. 304 pp., illus., + plates. $35.

Hummingbirds rival the F-16 in nonstop flight range and navigational accuracy but are cheaper to operate and ecologically beneficial as well. The hummingbird family, the Trochilidae, is one of the four largest bird families, 342 species by Johnsgard's count. Hummingbirds originated in the tropics of the Western Hemisphere. The 23 species that breed in North America pollinate at least 161 species of plants. Most of the hummingbirds in nine species breeding north of Mexico are migratory; the greatest distance traveled is about 5000 kilometers from Mexico to coastal Alaska by some populations of the rufous hummingbird.

Manifestations of interest in Audubon's "glittering fragments of the rainbow" range from casual observation of hovering at sucrose-water feeders to exploitation of their unique characteristics as subjects of basic biological research. The problems that extremely small body size presents for homeostasis are confronted daily by hummingbirds. Their feeding habits are quantifiable, most of their energy coming from "plants that