Number Theory Problem Is Solved

Mathematicians are elated by a surprising proof of a famous and important conjecture

An outstanding problem in mathematics—one that has baffled investigators for more than 60 years—has just been solved by a young German mathematician. "It is no exaggeration to say that, at least in number theory, this is the theorem of the century. It answers questions that seemed absolutely unanswerable," comments Spencer Bloch of the University of Chicago.

The result, which was derived by 29-year-old Gerd Faltings of Wuppertal University, is a resolution of the Mordell conjecture. In 1922, Lewis J. Mordell published a paper in the *Proceedings of the Cambridge Philosophical Society* in which he made an educated guess about the number of rational solutions to polynomial equations in two variables. Essentially, what Mordell conjectured was that a large class of polynomial equations only have a finite number of rational solutions. Faltings' solution to this problem is of such significance, according to Bloch, that "just the applications of his result will fill volumes."

Faltings told *Science* that although he thought he had "an outside chance" of proving the Mordell conjecture, he was nonetheless surprised when he succeeded. It took, he recalls, 18 months of work, mostly thinking and tinkering with a pencil and paper. When he thought he had a proof, he sent it out to a few mathematicians. The response was immediate. "At the moment, I'm getting very much response—much more than I am used to," Faltings says. Faltings' proof, a 40-page typed manuscript, is now being circulated in the mathematics community. Although it is considered to be an easy proof, "easy" is a relative term. Mathematicians who are not specialists in algebraic geometry, for example, would not expect to understand it without substantial effort.

Harvard mathematician Barry Mazur says that Mordell's conjecture bears on what might be termed the riddle of algebra. If you have a polynomial equation, such as $y^2 + y = x^3 - x$, and the equation has rational coefficients, meaning that any constant terms are ratios of integers, what can you say about the rational solutions, meaning solutions where x and y are ratios of integers? Are there any such solutions? If so, how many? Can you find them? Does the set of solutions have any mathematical structure that might enable you to find it and look within it for particular solutions? These questions are of such central importance in mathematics; Mazur says that "a vast number of diverse problems fall under its rubric."

Mathematicians classify equations according to the number of variables in them. Equations in only one variable have only a finite number of rational solutions. For example, the equation $x^2 - 1 = 0$ has only two rational solutions: 1 and -1.

But equations in two variables, the so-called algebraic curves, are more interesting. (Whenever there are two variables, the graph of the equation in ordinary two-dimensional space is a curve.) Sometimes it is obvious that there are an infinite number of rational solutions. For example, the equation $x + y = 0$ has as its solution $y = -x$, which is a straight line with an infinite number of rational points. Other times, it is not at all clear how many rational solutions there are.

To get at this problem of finding solutions for polynomial equations, mathematicians have resorted to topology—the study of surfaces. "It may seem strange, but to do arithmetic, you first have to do some geometry," Mazur remarks. Every algebraic curve has associated with it a geometric surface. This surface is usually made up of all the complex-valued solutions to the equations.

Topologists classify these surfaces according to the number of holes in them. The complex-valued solutions to equations of degree 1 or 2, like $x + y = 0$ or $x^2 + y^2 = 1$, make up a surface with no holes—like the surface of a ball. These curves either have no rational solutions or they have an infinite number of them. And if they have an infinite number of them, there is a rule to systematically enumerate all of them once you find one.

The solutions to elliptic curves, which are polynomials of degree 3, make up a topological surface with one hole—like the surface of a doughnut. For example, $x^3 + y^3 = 1$ is such a curve. "Degree 3 is a swing case," says Bloch. "Elliptic curves are not fully understood. We think we know the answer but we can't prove it."

All curves whose associated topological surfaces have two or more holes fall into a third category. Mordell conjectured that any curve in this category has only a finite number of rational solutions. Faltings proved this conjecture and, more generally, he proved that these curves have only a finite number of solutions in any given number field. For example, instead of looking for rational solutions, you could look for solutions in the form of a rational number plus $\sqrt{2}$ times another rational number. There would only be a finite number of these solutions also.

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The sorts of equations whose associated surfaces have at least two holes are, generally, equations of degree at least 4. The most famous of these equations are the so-called Fermat curves. In the 17th century, the French mathematician Fermat scribbled a note in the margins of a book saying that he had a proof that equations of the form $x^n + y^n = 1$, where n is at least 3, have only two or four rational solutions depending on whether n is even or odd. But no one has been able to reconstruct Fermat's proof if, in fact, he had one. According to Ronald Graham of Bell Laboratories in Murray Hill New Jersey, mathematicians have been able to calculate on computers that the conjecture is true for *n* up to 100,000, but they do not know whether it is true in general.

The surfaces associated with the Fermat curves have two or more holes. So, according to Mordell's conjecture, they must have only a finite number of rational solutions. This, of course, does not prove that the only rational solutions are zero, but it certainly says a lot more about the solutions to the Fermat equations than ever was known before.

Other equations that, according to the Mordell conjecture, must have a finite number of rational solutions, can be more complicated. Mathematicians have a formula to decide how many holes are in the geometric surfaces associated with the equations, and the evaluation of the formula depends on how smooth the geometric surfaces are. The Fermat curves, for example, have perfectly smooth surfaces associated with them. All nonsingular equations of degree four are associated with surfaces with two holes. Nonsingular equations of degree five are associated with surfaces with five holes.

Equations that have singularities are associated with surfaces that have rough spots. To calculate the number of holes in these surfaces, mathematicians count up the number of singularity and rate each singularity according to how bad it is. Then they subtract a correction term that accounts for these singularities from the formula telling how many holes there would be if the surface were completely smooth.

In proving the Mordell conjecture, Faltings drew on a large body of work in algebraic geometry, particularly results of John Tate of Harvard University and the Russian mathematicians I. R. Shafarivich, A. N. Parsin, Y. G. Zarkin, and S. Arakelov. This work involved Abelian varieties, which are higher dimensional algebraic spaces defined over number fields. "The Russians showed that the Mordell conjecture follows from specific facts about Abelian varieties and experimented with a 'theory of heights.' Faltings saw very clearly what to do with that theory and how to do it," Mazur says.

Although Faltings did not have to develop new techniques to solve the Mordell conjecture, mathematicians nonetheless are enormously impressed that he was able to do it. "It's a sensational result. Many people spent lots of time working on that problem," says Graham. "It's a wonderful piece of work," says Mazur. "Nobody thought the tools were there to resolve the conjecture and no one thought anyone could ever solve it," says Bloch.

Now mathematicians plan to study the proof and its implications in great detail. Those who have had a chance to study the proof will be giving courses on it in the fall. Bloch, who will be teaching such a course, remarks that he will have a great deal to say. "I hope I can do it in a year. The proof doesn't look that hard but the ramifications are enormous," he says.—GINA KOLATA

Fermilab Energy Saver Hits 500 GeV

Most of the high-energy physics news has been coming from Europe lately, but the momentum may be starting to shift back across the Atlantic. On 3 July, the Fermi National Accelerator Laboratory (Fermilab), while testing its new superconducting proton synchrotron, reached a record energy of 512 billion electron volts (GeV). Although this energy is a scant 7 GeV above the previous mark held by the laboratory's original accelerator, the achievement is a major breakthrough. The new machine, temporarily dubbed the energy saver because its superconducting magnets consume less power than conventional electromagnets, conclusively demonstrates that physicists have mastered the complex technology of superconductors and thereby paves the way for the next generation of ultrahigh-energy proton synchrotrons. The energy saver itself is destined to be transformed into the Tevatron, when it reaches 1 trillion electron volts (TeV) in the near future.

Starting up a new accelerator is always a nerve-wracking business because the mathematical equations that describe the orbits of the accelerated particles are not exactly soluble. Physicists are therefore at least a little uncertain that their machine will work until it is turned on. In Fermilab's case, the new superconducting technology made the apprehension even worse. A counterbalancing source of confidence was the ability to make unusually detailed simulations of the orbits of the proton beam because the characteristics of each magnet was stored in a computer. So far, says J. Richie Orr, the head of Fermilab's accelerator division, the simulated and actual behaviors have been nearly identical. "The champagne went up for grabs when we hit 512 GeV," adds Orr.

The first round of experiments is scheduled for this fall and will run at a modest 400 GeV. Before that time, two things need to be accomplished. One is to increase the beam current. All the tests so far have been with short pulses of 3×10^{10} protons with the pulses coming at intervals of about 20 seconds. Fermilab has promised experimenters they will have from 1×10^{13} to 3×10^{13} protons per pulse. The second thing is to learn how to extract the protons from the synchrotron and guide them to the numerous experiments. Buoyed by the success of the computer simulations in predicting machine performance, Orr expects no difficulty with the extraction at 400 GeV. A bigger problem will come in extracting a 1-TeV beam. As few as 10^9 protons per square centimeter that are lost from a beam of this energy during this process and that bombard the superconducting magnets could cause them to quench (lose their superconductivity), which halts experiments.

Experiments at 1 TeV await the 1985 upgrading of the experimental areas (radiation shielding, improved detectors, and so on). A year after that, Fermilab hopes to be able to collide a 1-TeV beam of protons against a 1-TeV beam of antiprotons. The W and Z^0 that were so proudly found at the European Laboratory for Particle Physics (CERN) earlier this year with its 270×270 GeV, nonsuperconducting proton-antiproton collider would be produced in much greater numbers at the higher energy, allowing more detailed examination. Other particles, predicted and unpredicted, presumably await, as well.

The leap into superconducting accelerators, a gamble undertaken alone by American physicists, had paid off. "The machine was so speculative, it is really a good feeling to see it work the way it is supposed to," concludes Orr.

—ARTHUR L. ROBINSON