The Quantized Hall Effect

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Summary. Quantization of the Hall effect is one of the most surprising discoveries in recent experimental solid-state research. At low temperatures and high magnetic fields the ratio of the Hall voltage to the electric current in a two-dimensional system is quantized in units of h/e^2, where h is Planck's constant and e is the electronic charge. Concomitantly, the electrical resistance of the specimen drops to values far below the resistances of the best normal metals.

The Hall effect is one of the better understood physical phenomena and is widely used in semiconductor materials laboratories to determine the carrier concentration of a given specimen. A magnetic field of moderate strength, an electric current supply, and a voltmeter are sufficient to perform combined Hall and resistivity measurements, which can yield direct information on the basic electrical properties of a new material. From an effect seemingly so well understood and a measurement as routinely performed as this, one hardly expects any surprises. And yet, less than 3 years ago, a startling observation created a new interest in the physical principles underlying the Hall effect. Von Klitzing *et al.* (*1*) discovered that under certain conditions the Hall resistance of their specimen was surprisingly constant, and its magnitude coincided with the ratio of two fundamental physical constants to any accuracy to which they were able to measure the effect (see Fig. 1). The Hall resistance R_H (see Fig. 2) was found to be quantized to

$$R_H = \frac{h}{ie^2} \tag{1}$$

where h is Planck's constant, e is the electronic charge, and the quantum number i (= 1,2,3, . . .) is the number of fully occupied quantum energy levels (the Landau levels). This result not only attracted the attention of solid-state physicists, experimentalists as well as theorists, but also stirred much interest in disciplines as distant from solid-state science as elementary particle physics. It presents the possibility of a quantum resistance standard in terms of fundamental physical constants and also a new method for determining the fine structure constant, which is a measure of the coupling between elementary particles and the electromagnetic field. The fine structure constant, α, can be related to the quantized Hall resistance by

$$\alpha = \frac{\mu_0 c}{2}\frac{e^2}{h} = \frac{\mu_0 c}{2}(R_H)^{-1} \tag{2}$$

where μ_0 is the permeability of the vacuum and by definition equals $4\pi \times 10^{-7}$ H/m. Since the light velocity, c, is known very precisely, the quantized Hall effect immediately spurred speculations that it could provide a new solid-state determination of α with an accuracy higher than that of previous determinations. At this time an accuracy of 1.7 parts in 10^7 has already been achieved (*2*). This is comparable to the accuracy of earlier measurements based on different physical phenomena, and further improvement is expected.

The quantized Hall effect is observed under conditions that are uncommon compared to those of standard Hall measurements. Magnetic fields of approximately 100 kilogauss and temperatures close to absolute zero are required. The specimen, too, is exceptional. It contains a so-called two-dimensional electron gas, which is ultimately responsible for the occurrence of this new quantum effect. The active region of all metal oxide semiconductor field-effect transistors (MOSFET's) consists of such a two-dimensional electron gas. In these systems the carriers are confined to a very narrow region at the interface between two different materials; they are able to move freely along the plane of the interface but lack any degree of freedom normal to it. Being confined to a narrow well of approximately 10^{-6} cm, they are quantum mechanically bound to the interface. In Si-MOSFET's, the two-dimensional electron gas exists at the interface between a slab of crystalline silicon and a thin (~ 10^{-5} cm) amorphous silicon dioxide top layer. The carriers are kept at the interface by a strong electric field established by an external voltage (gate voltage) applied to a metal electrode (gate) which covers the oxide. The Si-MOSFET is the most common physical realization of the concept of a two-dimensional electron system, and its fundamental properties have been studied extensively during the past two decades. The quantized Hall effect was first observed in such a device.

More recently, a new structure became feasible which, in various respects, has proved to be a superior host material for two-dimensional electrons (*3*). The structure is called a modulation-doped GaAs-(AlGa)As heterojunction, and it is prepared by a highly sophisticated crystal growth technique termed molecular beam epitaxy (*4*). It resembles the metal oxide semiconductor structures, but in this case the electron gas exists at the highly perfect interface between two crystalline semiconductors. The GaAs-(AlGa)As interface provides a much smoother background for the in-plane motion of the electrons. Furthermore, the binding electric field is not established by an external voltage, as in the MOSFET but is generated internally through positively charged centers within the (AlGa)As. Shortly after the discovery of the quantized Hall effect in Si-MOSFET, the same phenomenon was observed in GaAs-(AlGa)As structures by Tsui and Gossard (*5*) (see Fig. 3). The effect could be observed at higher temperatures and lower magnetic fields, making the experimental requirements less stringent than in the case of the Si-MOSFET. Hence a good fraction of present studies are performed on GaAs-(AlGa)As materials.

Apart from high-precision measurements of R_H, a considerable amount of experimental and theoretical work (*6–14*) has been devoted to unraveling the physical principles underlying the phenomenon. The most startling recent observation concerns the resistivity of the specimen under study. It is found that, under conditions where the Hall resistance is quantized to any of its values $R_H = h/ie^2$, the resistivity ρ_{xx} (see Fig. 2) of the two-dimensional electron gas appears to

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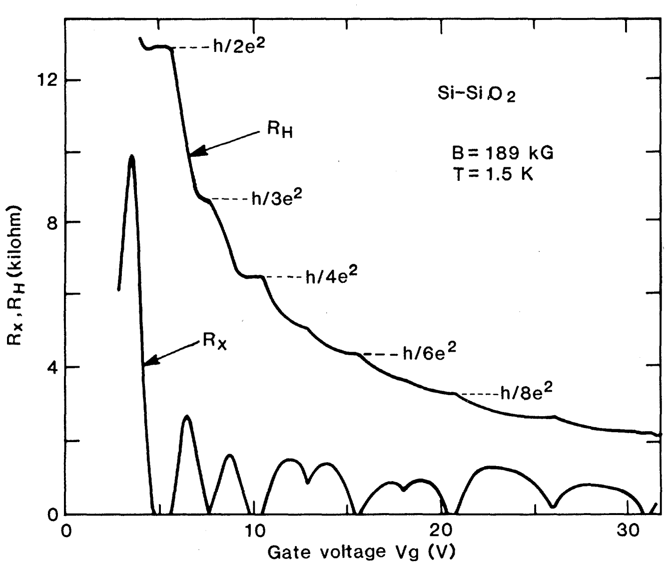
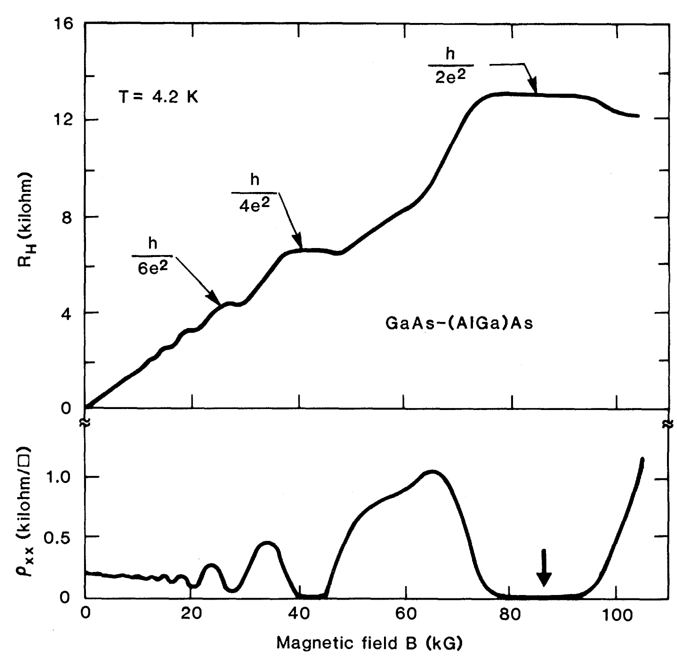
vanish as the temperature is lowered. Two-dimensional resistivities as low as $< 5 \times 10^{-7}$ ohm per square, equivalent to three-dimensional resistivities of $< 5 \times 10^{-13}$ ohm-centimeter have been reported at 1.23 K (*7*). This value is almost ten times lower than the resistivity of any nonsuperconducting material at any temperature, and it drops further when the temperature is reduced. Extrapolation to zero temperature indicates that a two-dimensional electron gas in a suitably high magnetic field is resistanceless; it represents an ideal electrical conductor, very similar to a superconductor, yet the phenomenon is caused by a completely different mechanism.

The beauty of the quantized Hall effect is that it represents the observation of such a fundamental relation in a field of physics seemingly so well understood. With a superficial glance at the system and a minimal understanding of two-dimensional transport, one may at first think the effect falls right into place. But there are many hidden complications that turn the quantized Hall effect into a scientific puzzle. Indeed, the recent discovery of fractional quantization with i = 1/3 and 2/3 defies theoretical explanation. In order to transmit some of the flavor of this puzzle, we start with a classical description to clarify some aspects of the problem and then discuss the effect in the case of integral quantization.

The Classical Two-Dimensional Electron Gas

The motion of electrons in crossed magnetic and electric fields evades intuition. In order to illustrate some of their properties, we consider the geometry shown in Fig. 4. The motion of the electrons is restricted to the x-y plane without friction or scattering. In the absence of an electric or magnetic field, all electrons move in straight lines in the plane, and since the direction and the speed are completely random, there is no net electrical current. This monotonous situation changes the moment an electric field E is applied to the system. If E is pointed in the negative x direction in the plane, each electron will accelerate in the positive x direction and, in the ideal frictionless case, will speed up indefinitely. In real systems, electrons are scattered by imperfections or vibrations of the atoms, leading to a motion analogous to that of a particle in a viscous fluid. After a very short initial acceleration time, the system approaches a steady state with a constant drift velocity, v_D, which for small E is proportional to E. The resulting current density is given by $j = env_D$, where n is the average number of electrons per unit area. In macroscopic terms, the relation between E and j is characterized by either the





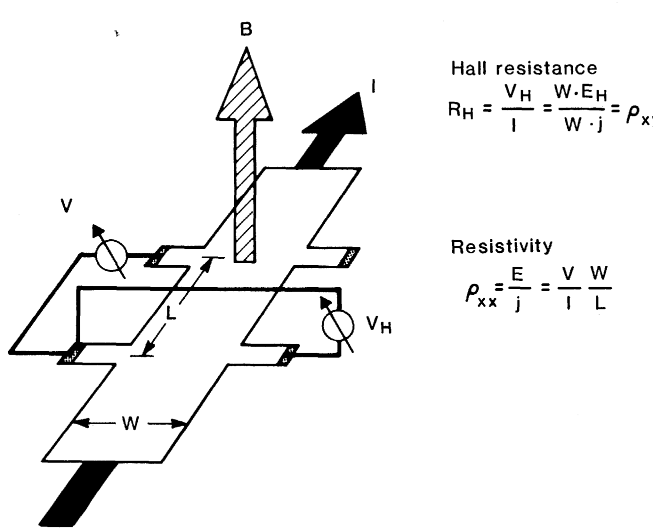


Fig. 1 (top left). The quantized Hall effect in a Si-MOSFET (*1*) in which the electron density is varied by a gate voltage V_g. Instead of being a smooth curve, the Hall resistance R_H develops plateaus having values h/ie^2, where i is an integer, and the resistance R_x of the specimen drops to very low values. Fig. 2 (bottom left). Schematic representation of a Hall experiment. The magnetic field B is perpendicular to the plane of the specimen and to the current I. The Hall resistance R_H and the resistivity ρ_{xx} are determined through the equations shown in the figure. Fig. 3 (above). The quantized Hall effect in GaAs-(AlGa)As heterojunctions (*5*). The electron density is fixed and the magnetic field is swept to exhibit the effect. At the arrow at 84 kG, ρ_{xx} is $< 5 \times 10^{-7}$ ohm/□.

conductivity σ or the resistivity ρ of the system through $j = \sigma E$ and $\rho = 1/\sigma$. The transport coefficient σ is given by $\sigma = env_D/E = en\mu$, where the mobility, $\mu = v_D/E$, describes the degree to which the carriers are able to move through the system.

Adding a magnetic field B to the system changes the situation considerably. In the absence of E, a B field in the z direction—that is, perpendicular to the electron plane—exerts a constant force $F = evB$ on an electron traveling with speed v. The direction of this Lorentz force is perpendicular to the direction of motion of the electron and perpendicular to the direction of B. As a result, the electron executes a rotating motion in the plane on a circle with a radius $r = mv/eB$, where m is its mass, and with a frequency (the cyclotron frequency) $\omega_c = v/r = eB/m$. Since the B field does not change the speed of the electrons, their energy remains independent of B and can be expressed as $\epsilon = 1/2m\omega_c^2r^2$. Thus, an ideal two-dimensional electron system in a magnetic field can be visualized as a system of electrons rotating with a constant frequency ω_c around the field lines on circles having radii proportional to the speed of the electrons (Fig. 5).

The addition of an electric field E affects the electron system quite differently than in the absence of B. Instead of drifting along the x direction, the carriers move in the direction that is perpendicular to the E and B fields—the y direction (Fig. 6). Each electron keeps rotating while the center of its rotation is drifting aside. This, again, is a result of the Lorentz force: the E field accelerates the electron in the x direction while the B field deflects the motion into the y direction. In contrast to the ideal frictionless case in the absence of B, the carriers are not accelerated indefinitely. The centers of their orbits move with a constant velocity $v_B = E/B$ parallel to the y direction. Therefore, the entire electron system drifts aside with a constant velocity in the direction perpendicular to E and B, representing a constant current in the y direction. The current density is given by $j = env_B = enE/B$. However, the current and the electric field are not parallel, as in the absence of a magnetic field, but perpendicular to each other. The current parallel to the E field is zero.

A description of this behavior in macroscopic terms, which requires two independent transport coefficients, leads to some surprising results. The conductivity, σ_{xx}, describing the current density along the electric field is zero. However,

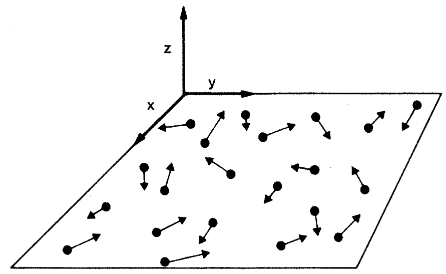


Fig. 4. Schematic of an ideal two-dimensional electron system where the electrons with random speed and direction are confined to move in the x-y plane.

the resistivity, ρ_{xx}, defining the electric field strength along the current path also vanishes, since there is no E-field component along the current. We encounter an exceptional situation where the conductivity and the resistivity vanish simultaneously. This striking result is induced by the magnetic field, which diverts the current from the direction of the applied electric field. In other words, the current and the electric field are mutually orthogonal and the conduction is free from dissipation. The Hall conductivity, σ_{xy}, and the Hall resistivity, ρ_{xy}, relating E and j through $j = \sigma_{xy}E$ or $E = \rho_{xy}j$, are given by

$$\sigma_{xy} = \frac{ne}{B} \tag{3}$$

and

$$\rho_{xy} = \frac{B}{ne} \tag{4}$$

We note that in two dimensions R_H is identical to ρ_{xy} and $R_H = \rho_{xy} = B/ne$ (see Fig. 2).

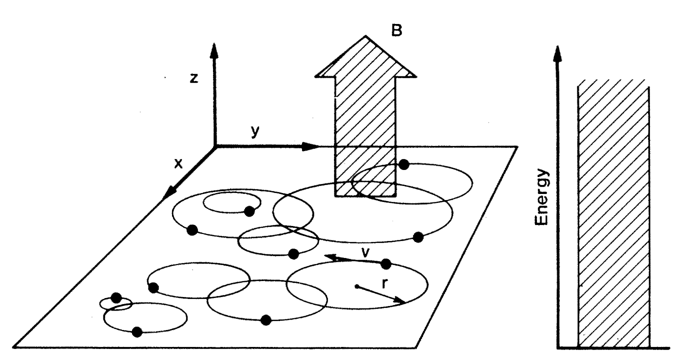
The Hall conductivity σ_{xy} and resistivity ρ_{xy} are unusual in that they relate current in one direction with an electric field pointing perpendicular to it. The usual parallel conductivity σ_{xx} and resistivity ρ_{xx} vanish completely. This last fact has important conceptual consequences. If we consider the x direction alone our system is an insulator since, in spite of the application of an electric field, there is no current flow along this direction. On the other hand, if we consider the y direction, the system should be termed an ideal conductor. Although a constant current is flowing, no electric field along this direction is necessary to support it and consequently, as in a superconductor, no dissipation of electric power accompanies the steady current. However, one cannot be too surprised about this result since the ideal model system excluded any kind of friction. Even in the absence of a magnetic field, such a system would appear to be without dissipation. In real systems, where electrons scatter at vibrating atoms or imperfections of the material, leading to a finite amount of friction, the values for σ_{xx} and ρ_{xx} generally deviate from zero. However, we will find that this ideal case can be realized in real two-dimensional systems under certain conditions.

Introduction of Some Quantum Mechanics

The inhibition of electron scattering in a real two-dimensional system in a high magnetic field is due to principles that are beyond our classical description. They require the introduction of some fundamental rules of quantum mechanics. The laws of quantum mechanics will not only bring about vanishing resistance of a real two-dimensional system in a magnetic field, but will also be responsible for the discontinuous behavior of the transport coefficients σ_{xy} and ρ_{xy} that led to the observation of the quantized Hall resistance and, in turn, to the high-precision determination of the fine structure constant. The following paragraphs introduce the quantum mechanical rules that are pertinent to the problem of electrons in a magnetic field.

The fundamental difference between a quantum mechanical and a classical treatment of an electron in a magnetic field is that only a discrete set of orbits is

Fig. 5. Classical motion of a two-dimensional electron system with a magnetic field (B) normal to the plane. The energy of the carriers is unaffected by the field and remains $\epsilon = 1/2\ mv^2$; hence all energies are possible.



accessible to the electron in the former case. Electrons can occupy only discrete states with well-defined discrete energies. The allowed radii for electron orbits in a magnetic field are the so-called Landau radii

$$r_l = \left[\frac{2\hbar}{eB}\,(l - 1/2)\right]^{1/2} \tag{5}$$

where $\hbar$ is an abbreviation for $h/2\pi$ and the quantum number l can be any positive integer $l = 1,2,3\ldots$ (see Fig. 7). Since their orbits are quantized, the energies of the electrons form a sequence of Landau levels, given by

$$\epsilon_l = 1/2m\omega_c^2 r_l^2 = (l - 1/2)\hbar\omega_c \tag{6}$$

where $\omega_c \equiv eB/m$ is the cyclotron frequency.

Finally, electrons have to obey Pauli's exclusion principle that no two electrons can agree in all their quantum numbers. The exclusion principle in effect limits the number of electrons per unit area that can occupy each Landau level. This number is the degeneracy of each Landau level and is given by

$$s = \frac{eB}{h} \tag{7}$$

At very low temperatures electrons will occupy the allowed states with the lowest energy. In a given magnetic field a system of two-dimensional electrons with density n will arrange itself in the following way. Of the n electrons per unit area, s will occupy the energetically lowest Landau level $l = 1$, each having an energy $\epsilon_1 = 1/2\hbar\omega_c$ and an orbit with a radius $r_1 = (\hbar/eB)^{1/2}$. The same number s will occupy the next higher level $l = 2$, having energies $\epsilon_2 = 3/2\hbar\omega_c$ and radius $r_2 = (3\hbar/eB)^{1/2}$. Loosely speaking, they form a second layer, although they actually reside within the same plane and "layer" is to be understood in energetic terms. All electrons can be accommodated by filling consecutive Landau levels. The last level generally will remain partially unoccupied since n generally is not an integral multiple of the degeneracy s. The Fermi energy (ϵ_F) is the energy of the last electron accommodated in the system at absolute zero temperature. It may be regarded as the energy that divides the filled and the empty levels of the system.

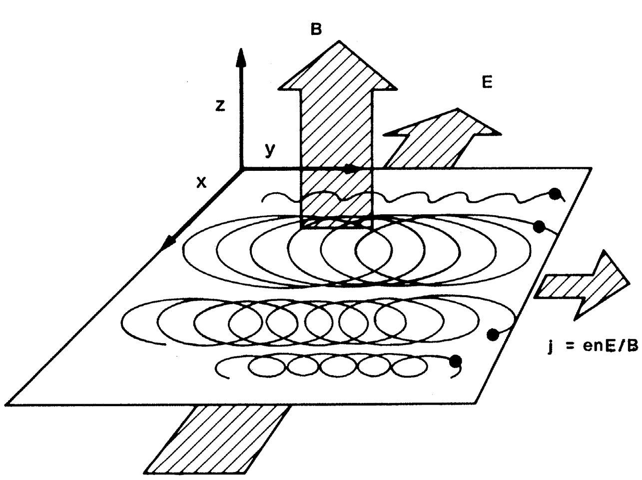
The important point to notice is that, in distributing the electrons over the levels, an abrupt break occurs whenever one Landau level is completely filled. This is due to the fact that an additional amount $\hbar\omega_c$ of energy is required to accommodate each electron in the next higher level. In the region where an integral number l of Landau levels is just filled ($n = l \cdot s = l\ eB/h$) a slight variation of n (or B) will drastically change the energy of the system. These jumps in energy, which do not occur in a classical treatment, have important consequences for the scattering of electrons in a real two-dimensional system, where a finite amount of scatterers is always present.

Electron Scattering in a Quantized System

An electron encountering any kind of a defect center will be scattered out of its orbit (initial state) into a new orbit (final state) and may lose or gain energy in the process. Such a scattering event can occur only if empty orbits are available for the electron to be scattered into. In our quantized two-dimensional system, since electrons can only assume discrete energies $\epsilon_l = (l - 1/2)\hbar\omega_c$, energetic exchange between scatterers and electrons is limited to multiples of $\hbar\omega_c$ (for inelastic events) or 0 (for elastic events). At low temperatures and high magnetic fields, when the Landau level splitting $\hbar\omega_c$ vastly exceeds all thermal energies, only elastic events, scattering of electrons among orbits within the same Landau level, are feasible. The scattering is therefore limited by the number of empty orbits within the same Landau level.

Total suppression of scattering occurs when all orbits of the occupied Landau levels are completely filled and all higher Landau levels are completely empty—that is, when the Fermi energy resides somewhere within the gap between two subsequent Landau levels. In this case no scattering can take place, since the empty orbits in the higher Landau levels are inaccessible to electrons in the completely filled Landau levels. Therefore, the complete occupation of an integral number of Landau levels leads to vanishing electrical resistance. We emphasize that the realization of this zero resistance does not require the absence of scatterers within the two-dimensional system, it requires the absence of possibilities for the electrons to scatter. In this way, the real two-dimensional system mimics the ideal model system, creating a state with vanishing resistance, $\rho_{xx} = 0$, in spite of the existence of scatterers.

In certain respects this zero-resistance state is similar to superconductivity. In both cases it is the existence of a finite gap in the energy spectrum, with all states below the gap occupied and all states above the gap unoccupied, which leads to vanishing electrical resistance. Nevertheless, the gaps are of very different origin, and various properties, like magnetic field exclusion in superconductors and the existence of the quantized Hall effect in two-dimensional systems, are not common to both phenomena.



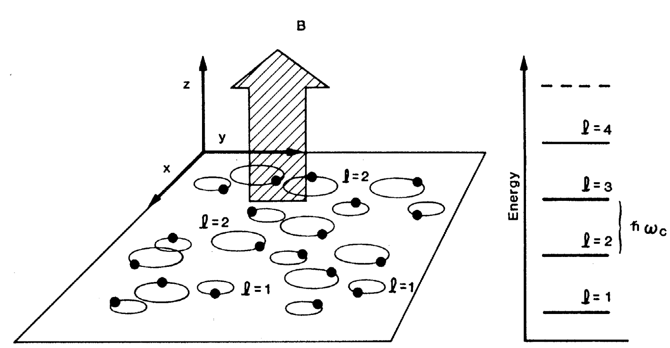


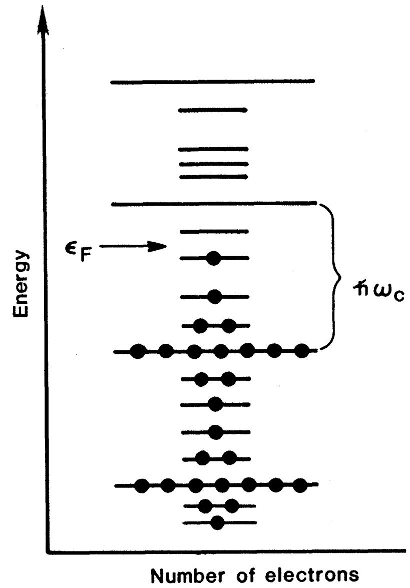
Fig. 6 (left). Classical motion of carriers of a two-dimensional electron system in crossed magnetic (B) and electric (E) fields. While rotating, the carriers drift into the direction normal to B and normal to E, giving rise to a current density $j = enE/B$ in the y direction. Fig. 7 (right). Quantum mechanical motion of carriers of a two-dimensional electron system in a perpendicular magnetic field B. Only certain orbits and their energies are allowed, as given in Eqs. 5 and 6 in the text. (Radii with $l > 2$ are omitted for clarity.)

Integral Quantization of the Hall Resistance

We have seen that, because of the discrete nature of its quantum mechanical energy spectrum, a real two-dimensional electron system can behave as our ideal classical model system. It can carry an electrical current without dissipation (that is, $\rho_{xx} = 0$) when an integral number, i, of Landau levels are completely filled. Under this condition, the total density of electrons has to remain $n = is = i\ eB/h$, where $s = eB/h$ is the degeneracy of each Landau level. The Hall resistance R_H, which in two dimensions is the same as the Hall resistivity ρ_{xy}, is then given by $R_H = \rho_{xy} = h/ie^2$, exactly as observed experimentally by Von Klitzing *et al.* (*1*) and Tsui *et al.* (*2*).

However, our discussion, which is based on a perfect two-dimensional system, also precludes the experimental observation of this quantum phenomenon. It does not provide the means to keep the filled Landau levels completely occupied for an extended range of either the electron density or the magnetic field, which is necessary for an experimental observation in the form of Hall plateaus (see Figs. 1 and 3). Interestingly, the existence of imperfections in the samples is essential for the observation of the quantized Hall effect.

Imperfections in the two-dimensional system give rise to the states that can trap electrons. The trapped electrons do not contribute to the electrical current and are referred to as localized electrons, setting them apart from the current-carrying delocalized electrons. Depending on the strength of the localizing potential, the energies of localized electrons deviate more or less from the quantized energies of the delocalized electrons and consequently are found somewhere within the gap region between the Landau levels. The modified energy spectrum of a real two-dimensional system therefore consists of Landau levels representing the delocalized orbits and a broad distribution of localized orbits filling the gaps in between (see Fig. 8). The existence of localized orbits buffers the abrupt jumps of the Fermi level from one Landau level to the next which would occur in the absence of these gap states. As long as a variation in density or in magnetic field adds electrons to or subtracts electrons from consecutive localized orbits, the Fermi energy resides within the gap region between Landau levels and the number of delocalized orbits remains unaltered for an extended range of electron density or magnetic field. Since only delocalized orbits contribute to the electric current, the transport properties of the system remain constant as long as E_F resides in the gap, leading to the occurrence of plateaus in ρ_{xy} and zero in ρ_{xx}. In this sense it is the imperfection of a real two-dimensional system which brings about the plateaus as a signature of the quantized Hall effect.



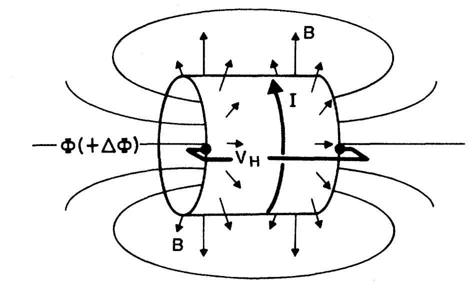


Fig. 8 (left). Energy spectrum of a two-dimensional electron system in a perpendicular magnetic field including electron localization. The energies of localized orbits (on short lines) are found in the gaps between Landau levels (long lines). The Fermi energy ϵ_F resides between Landau levels. Fig. 9 (right). Geometry for Laughlin's gedankenexperiment. The two-dimensional electron system forms a cylinder. A strong magnetic field B pierces it everywhere normal to its surface. A current I circles the loop, giving rise to a Hall voltage V_H and a small magnetic flux Φ along the axis of the cylinder.

However, this scenario is unable to account for the high accuracy to which the values of the plateaus are quantized. Since a given fraction of the carriers are localized, the density of delocalized electrons is diminished accordingly and the number of current-carrying electrons in each Landau level deviates considerably from its ideal value of $s = eB/h$. Hence ρ_{xy} is expected to deviate accordingly from its quantized value $\rho_{xy} = h/ie^2$.

Some light was shed on this puzzling situation by Prange (*8*) and Aoki and Ando (*9*), who calculated the current of an ideal two-dimensional system in a magnetic field containing an isolated scatterer which traps one electron, thereby removing it from the current-carrying electrons. They obtained the startling result that the remaining electrons make up in current for the localized electron, which they skirt, by increasing their own velocity. The situation is analogous to the flow of an incompressible fluid circumventing an obstacle and increasing its speed at the position of the bottleneck in order to keep the current constant (*15*).

Laughlin's Explanation

A very elegant gedankenexperiment by Laughlin (*10*), which was extended by Halperin (*11*), treats the quantized Hall effect from a very general point of view and arrives at the correct answer, independent of the physical details of the system. Their gedankenexperiment requires the notion of gauge invariance, a physical symmetry beyond common intuition, and we will outline the basic ideas underlying their arguments.

Laughlin based his consideration on an unusual, but feasible, geometry (see Fig. 9). The two-dimensional electron system is bent to form a cylinder whose surface is pierced everywhere by a strong magnetic field B normal to the surface. An arbitrary current I is assumed to circle the loop. As described earlier, the action of the magnetic field on the charged carriers gives rise to a voltage V_H perpendicular to the current—that is, from one edge of the cylinder to the other. As a result of this circulating current, a small magnetic field threads the current loop, giving rise to a magnetic flux Φ through the cylinder. The aim of the gedankenexperiment is to establish the relation between I and V_H.

To determine I, we use an electromagnetic equation

$$I = \frac{\delta U}{\delta \Phi} \tag{8}$$

which relates I to the total energy, U, of the electronic system, which is free of dissipation, and the magnetic flux, Φ, piercing the current loop. The value of I can then be established by a slight variation, $\delta\Phi$, of the magnetic flux and simultaneous determination of the change in the total electronic energy, δU, of this system. The carriers are separated into two distinct classes: localized electrons, which are excluded from the transport of current, and delocalized electrons, which encompass the loop. The two groups react quite differently to $\delta\Phi$. Localized electrons remain totally unaffected, as one would expect, since there is no change in magnetic field at their positions and they do not enclose any fraction of Φ. Delocalized electrons, which enclose Φ, experience the flux change and generally do change their energy.

Since $\delta\Phi$ is too small to transfer electrons between Landau levels, its only effect is to move the electron orbits of the same Landau level within the surface of the cylinder. Any motion in the direction of the external electric field E established by the potential drop V_H will modify the electron energy by some amount δU. To determine the actual value of δU, Laughlin noticed that after the magnetic flux Φ is varied by a finite, though exceedingly small, amount of a flux quantum, $\Delta\Phi = h/e$, all electron orbits of the system are identical to those before the flux quantum is added. The distribution of electrons among the orbits might have changed during the process—for instance, electrons might have moved into other orbits, leaving empty orbits behind, or several electrons might have exchanged positions. Nevertheless, the orbits available to the carriers before the flux change are identical. For the general case of an arbitrary magnetic field, the change in orbit occupation is unknown and the evaluation of ΔU infeasible.

However, an exceptional situation develops when the Fermi level, ϵ_F, resides within the unaffected localized states. In this case, all delocalized orbits of all Landau levels below ϵ_F are completely filled, and excitation into a next higher Landau level is impossible because of the large amount of energy, $\hbar\omega_c$, required for such a transition. Since all accessible delocalized orbits were occupied before the addition of $\Delta\Phi$, all accessible delocalized orbits are occupied after the addition of $\Delta\Phi$, and all orbits before and after the change coincide, the total energy U of the system has to remain unchanged and $\Delta U = 0$. However, since one is unable to trace the motion of the electrons during the flux increase, one has to allow for the possibility that an integral number of electrons were transferred through the system during the flux change, entering the cylinder at one edge and leaving it at the opposite edge, without knowing their actual path. This electron transfer is the only way in which the highly degenerate two-dimensional electron system can vary its electronic energy. Moving from one edge of the cylinder to the other through the electrostatic potential V_H, an electron changes its energy by eV_H. If i electrons are transferred, the total change of the electronic energy is $\Delta U = ieV_H$, $i = 0,1,2,3, \ldots$. Returning to Eq. 5 and replacing the infinitesimal quantities by their finite equivalents, we find the current to be

$$I = \frac{\Delta U}{\Delta\Phi} = \frac{ieV_H}{e/h} = \frac{ie^2}{h} V_H \qquad (9)$$

and the Hall resistance, $R_H = V_H/I$, given by Eq. 1. Halperin (*11*) later identified the value of i as the number of occupied Landau levels.

The preceding discussion represents the present understanding of the origin of the quantized Hall effect. It shows that the existence of localized states is essential for the experimental observation. It is remarkable that a high-precision measurement should require the physical system to be imperfect, that the accuracy of quantum electrodynamics can be tested by an experiment resting on the localized states in a disordered system, and that the absence of electrical resistivity can be a consequence of the existence of imperfections.

Fractional Quantization

Very recently, investigations of GaAs-(AlGa)As heterostructures in magnetic fields as high as 200 kG and temperatures as low as 0.5 K revealed new surprises. In the so-called extreme quantum limit, when only the lowest Landau level is partially occupied, the quantum phenomena discussed above should not be present. Nevertheless, it has been discovered (*16*) that ρ_{xx} vanishes and ρ_{xy} is quantized in units of h/e^2 when occupation of the lowest Landau level is 1/3 and 2/3. This fractional quantization of the Hall resistance—that is, $R_H = h/ie^2$ with i = 1/3 and 2/3—differs from the integral quantization in that it is observable at lower temperatures and higher magnetic fields and is more pronounced in samples with higher electron mobility. These features suggest that the effect is more fundamental, and the search for an explanation of it is currently an active area of solid-state research. In short, the puzzle of the quantized Hall effect has not yet been entirely put together.

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