Continuation Methods: New Ways to Solve Equations

Mathematicians are finding that the intuitive ways to follow curves are not necessarily the best ways

Mathematical models of the economy are necessarily complex. Yet these models have, in theory, equilibrium points where all is in harmony and supply equals demand. Although such models have played a central role in economics research over the past quarter-century, they are fundamentally unsatisfying to many members of the field. The economists would like to know where these equilibrium points are and what paths an economy can take to get to them. But they cannot get this information without solving enormously complex systems of nonlinear equations—a task that has, until now, proved infeasible.

Recently, several mathematicians began developing new ways to solve such systems of equations. Their methods are applicable not only to mathematical economics but also to a wide variety of problems arising in fields such as seismology and fluid mechanics, providing results that one practitioner describes as "mind-boggling" and "flamboyant."

The new methods, called global continuation methods, involve carefully continuing along curves that lead to solutions of equations. Continuation methods themselves are familiar mathematical techniques, but researchers previously tried to follow curves only in an intuitive way. Frequently their methods worked, but the curves they followed also sometimes folded back on themselves, stopped, shot out to infinity, or began branching, making it difficult or impossible to follow them.

The problems with these old continuation methods were cracked when mathematicians realized that the intuitive ways to follow curves are not necessarily the best ways. They began using results from a branch of mathematics called topology to show that, in very general cases, almost all curves can be followed to solutions.

Topology, which is largely concerned with continuous deformations of mathematical surfaces or curves in higher dimensional spaces, has almost never previously been used to construct solutions to equations. Instead, it was used to show that solutions exist. For example, investigators used theorems of topology to show that mathematical models of the economy have equilibrium points.

These new methods are preceded by a curve-following technique discovered a decade ago by economists Herbert Scarf of Yale University and Curtiss Eaves, now at Stanford University. Their method relies on using an algorithm resembling linear programming to subdivide a space in search of a solution. Although the technique of Scarf and Eaves does not at first glance resemble curve-following, it can be interpreted in that way. It is also based on topological theorems, but of a slightly different sort than those used to devise the more recent methods. Scarf and Eaves use their method to solve complex systems of equations in economic models, but it is much more difficult to implement than the newer methods.

Some of the theoreticians behind the new methods are James Yorke and his associates at the University of Maryland, Stephen Smale and Morris Hirsch of the University of California at Berkeley, and Herbert Keller and his associates at the California Institute of Technology.

Yorke explains that he turned to topological methods because many topological proofs can be interpreted in terms of curve-following. He talks of mappings rather than equations (an equation can be described as a map from one region to another) and he looks for fixed points of the maps. Fixed points are points that are mapped to themselves. Showing that a map has a fixed point is equivalent to showing that the corresponding equation has a solution.

To find a fixed point of a particular map, Yorke and his associates start with a simpler map whose fixed point they know. Then, using topology, they continuously deform the simple map into the map in question. The curve representing the movements of the fixed point of the simple map as it is continuously deformed is described by a differential equation. Thus the investigators can follow this curve to a fixed point of their more complicated map.

Smale and Hirsch and Keller developed techniques that can easily be rephrased in the language of Yorke's techniques. These other investigators speak of finding zeros of functions rather than fixed points of mappings. But all three groups of investigators justify their work with the same topological tools. Using these tools, they demonstrated that when any one of a huge family of curves is followed it will not double back on itself, it will not dead-end, and it will not escape to infinity. And it will eventually approach some solution. It is still an open problem to get all possible solutions or even a particular one.

Smale, Hirsch, and Keller follow curves taking them from a zero of a simple function to a zero of a function that interests them by using a variation of a well-known approximation technique called Newton's method. Newton's method was previously used only when investigators knew they had a good approximation to a solution. The variation, however, lets researchers start almost anywhere in their search for the solution, provided they pay close attention to the curve they are following. The method also has the decided advantage that it does not stall at difficult points, called singularities, and it can be followed through branch points.

Keller and his associates are now using computer algorithms based on the new continuation methods to solve problems in fluid mechanics and seismology. Keller says that one of his more spectacular results has been solving a famous problem, the Karman swirling flow problem, involving laminar flows. The problem is to describe the fluid motion above a rotating infinite plane disk when the fluid at infinity does not rotate. Previously, investigators had computed only two flows. Keller, however, finds that there are infinitely many flows stacked up in layers—a result no one had ever dreamed of, he says.

Keller also used continuation methods to solve on a computer a nonlinear system of equations arising from earthquake predictions. These are equations describing seismic rays—vibrations in the earth radiating from an earthquake. In order to solve these equations, Keller first writes equations describing a ray connecting the earthquake source and the seismic receivers and having the proper number of reflections and transmissions. He then invents a fake material for which that ray would be correct. Finally, he uses continuation methods to slowly change from the fake material to the real material through which the ray travels. The advantage of these methods, Keller says, is that they allow him to calculate easily 20 to 30 rays in a model, whereas previously seismologists had trouble calculating two.

Smale is using continuation methods to find equilibrium points in economic models. Not only can he find these points, he says, but the computer algorithms he uses provide step-by-step paths to the equilibrium points. This result can provide insight into the fundamental problem of how economies can attain equilibrium.

The new continuation methods are still in their infancy. They seem to be applicable to an immense variety of problems and are easier to implement than previous continuation methods. Those who are developing the techniques are confident that they will become an important new tool.—GINA BARI KOLATA

Is Labile Hypertension a Myth?

Doctors are commonly taught to be on the lookout for so-called labile hypertensives. These are people whose blood pressure is high on one or several readings but drops to normal or to the borderline hypertension range on a subsequent reading. The conventional wisdom is that if a patient's lowest reading is normal, his doctor should not treat him, regardless of his other blood pressure readings.

This long-accepted view of labile hypertension is now being challenged by William Kannel, director of the Framingham study. The Framingham study is a 30-year-old longitudinal study of adults living in the town of Framingham, Massachusetts. It has long been a major source of information on the causes and consequences of cardiovascular diseases.

At last month's meeting of the American College of Cardiology in Miami, Florida, Kannel reported that there is no difference in terms of risk of heart disease, stroke, and congestive heart failure between labile and "true" or "stable" hypertensive patients. Kannel's report is now a subject of some controversy among hypertension specialists, a number of whom contend that it does not refute their belief that a group of truly labile hypertensive patients exists and that these patients are at lower risk than stable hypertensives.

The Framingham investigators obtained their results by analyzing data on blood pressure measurements of 5209 Framingham participants. These measurements were taken at 2-year intervals over a period of 20 years. When Kannel and his associates looked to see whether particular people are labile hypertensives every time they are examined, they found that there is a very low correlation of the lability of a person's blood pressure between one exam and another. But they noticed that lability is correlated with blood pressure—the higher the blood pressure, the more labile it is likely to be.

The explanation for their finding, Kannel says, is a statistical phenomenon called regression toward the mean. That is, the blood pressure of someone with severe hypertension is more likely to drop than to increase on a second reading. Conversely, the blood pressure of someone whose pressure is at the low end of the spectrum is more likely to increase than drop on a second reading. Because of this phenomenon, Kannel reports, one-third of hypertensive patients are classified as labile but not true hypertensives, and many are not treated who should be.

As further evidence that lability of blood pressure is a statistical artifact, the Framingham investigators note that when they analyzed their data to see how the risk of cardiovascular disease varies with lability of blood pressure, they found that the risk increases as lability increases. This would be expected if lability is more likely at higher blood pressures. When they adjusted their analysis for this effect, they found that labile hypertensives are at the same risk as stable ones.

Kannel suggests that doctors use the average of several office blood pressure measurements to determine a patient's blood pressure. Patients whose blood pressure is normal on one or another occasion should not be dismissed if their average pressure is high.

Although these recent findings about labile hypertension are unexpected, Kannel says he is not too surprised by them because they are so logical. Moreover, he is used to unexpected findings about high pressure. "Practically everything we investigate about high blood pressure goes against clinical teaching," he reports. "We used to think that only if the diastolic blood pressure was high the patients had problems. Then we found that the systolic blood pressure is just as important. We used to think hypertension in the elderly was less important than in young people. It turns out to be more important." (Older people with high blood pressure are at greater risk of cardiovascular disease). Thus, in Kannel's view, the myth about labile hypertension is just one more misconception that the Framingham study has exploded.

Kannel's view is challenged, however, by some experts on hypertension, including Edward Freis of the Veterans Administration Hospital in Washington, D.C. Freis explains that hypertensive patients are a heterogeneous population. Some who have labile blood pressure measurements in the doctor's office have perfectly normal blood pressures when the leave the office. These patients seem to be at lower risk of developing cardiovascular diseases. For example, Maurice Sokolow of the University of California at San Francisco found that these patients were less likely to have damaged body organs from high blood pressure. They are thought to have high blood pressure in the doctor's office as a response to their nervousness about seeing a doctor. Other patients with labile hypertension have high blood pressure measurements at home as well as in the doctor's office. Freis suspects that this second group of patients, who are probably at higher risk of developing cardiovascular diseases, is the basis of Kannel's finding. "You can't forecast for an individual on the basis of a group," he says.

Judging from Freis's reaction, the "myth" about labile hypertension has hardly been exploded. But it is likely to be reexamined, and some doctors will undoubtedly change their opinion that patients with labile hypertension should not be treated.—GINA BARI KOLATA