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Continuous and Discontinuous Perturbations

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Discontinuous functions are not unknown in the mathematical formulation or solution of physical problems; examples are idealized media boundaries and first-order phase transitions. Generally, in the context of solutions to problems, such as the formation of shocks in nonlinear wave propagation, certain details regarding the singularity depend on initial data or model parameters that enter the equations in fairly innocuous ways. Occasionally, the dependence on the model parameters is so smooth as to render the resulting discontinuity at first sight surprising. This appears to be the case with a class of examples in quantum theory that we shall describe in this article and that range from simple problems in quantum mechanics to extremely complex problems in quantum field theory that have so far resisted all conventional attempts at solution.

Summary. Perturbations of quantum systems ranging from oscillators to fields can be either continuous or discontinuous functions of the coupling. Even nonrenormalizable fields may now find a natural interpretation.

It seems self evident that if A and B are two quantities, then the sum $A + \lambda B$ is continuous in the parameter λ, and in particular, say, as $\lambda \to 0^+$ (decreases to zero from positive values) that $A + \lambda B \to A$. We propose to discuss counterexamples to this elementary proposition. Our examples are motivated by model problems in which $A = H_0$, the free Hamiltonian for some system, and $B = V$, the interaction potential, and what we consider then are examples for which $H_0 + \lambda V \not\to H_0$ as $\lambda \to 0^+$. This is not simply a discontinuity among states of a given Hamiltonian (as can happen when degeneracies occur), but a discontinuity of the Hamiltonian itself. Normally, as we shall see, there is convergence as $\lambda \to 0^+$; that is, $H_0 + \lambda V \to H_0'$ as $\lambda \to 0^+$, but $H_0' \neq H_0$. We emphasize that H_0', called a pseudofree Hamiltonian, generally has eigenvectors and eigenvalues different from those of H_0. Furthermore, one finds that $H_0' + \lambda V \to H_0'$ as $\lambda \to 0^+$. In this situation we say that V is a discontinuous perturbation of H_0, while V is a continuous perturbation of H_0'. Figure 1 is a pictorial representation of this general state of affairs. And to illustrate the general concepts, let us consider the following simple example.

Prototype Elementary Example (Field Theory in Miniature)

Take as the free Hamiltonian the standard harmonic oscillator for a single degree of freedom with amplitude x in the Schrödinger representation

$$H_0 = -\tfrac{1}{2} \frac{\partial^2}{\partial x^2} + \tfrac{1}{2}x^2 \qquad (1)$$

with unit angular frequency and mass, and the choice of dimensions is such that $\hbar = 1$ ($\hbar$ = Planck's constant/2π). The free spectrum of eigenvalues $E_n = n + \tfrac{1}{2}$, $n = 0, 1, 2, \ldots,$ and the usual eigenfunctions fulfill $\psi_n(-x) = (-1)^n\psi_n(x)$. Take as a perturbation

$$V = |x|^{-\alpha} \qquad (2)$$

the singularity of which depends on the magnitude of α. Even this elementary example exhibits interesting properties as a function of α (*1*). Broadly speaking, if $\alpha \leq 2$ the potential can be defined to be a continuous perturbation of the harmonic oscillator, while if $\alpha > 2$ this is impossible and the potential is necessarily a discontinuous perturbation of the harmonic oscillator. It is not difficult to determine the zero-coupling limit when $\alpha > 2$. In fact, for any $\alpha > 2$ it follows that $H_0 + \lambda V \to H_0'$, where H_0' is described either (i) by the differential operator $-\tfrac{1}{2}\partial^2/\partial x^2|_{\rm DBC} + \tfrac{1}{2}x^2$, where DBC denotes Dirichlet boundary conditions at the point of singularity [that is, $\psi(0) = 0$], or (ii) by the eigenfunctions $\psi_n'(x)$, which are equal to the odd-parity harmonic oscillator eigenfunctions continued for negative argument as either even or odd functions, and the eigenvalues E_n', which are doubly degenerate values of E_n corresponding to the eigenvalues of the odd-parity harmonic oscillator states (see Fig. 2). It is notable that only one pseudofree Hamiltonian H_0' arises for all $\alpha > 2$. Of course, the pseudofree Hamiltonian varies with the location of the singularity, namely if Eq. 2 is generalized to $V = |x - c|^{-\alpha}$ for arbitrary c. In this case the Dirichlet boundary conditions apply at $x = c$, and generally no eigenfunction or eigenvalue of the harmonic oscillator survives in H_0'. On reintroduction of the perturbation in these examples, the energy levels depart continuously from those of the pseudofree Hamiltonian, typically, for small λ, as $O(\lambda)$, $O(-\lambda \ln \lambda)$, and $O(\lambda^{1/(\alpha-2)})$ for $2 < \alpha < 3$, $\alpha = 3$, and $\alpha > 3$, respectively (*2*).

In this simple example one sees that the very presence of a suitably singular potential, once it has been introduced, leaves an indelible imprint on the sys-

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tem, leading to a permanent and irreversible change. It is plausible that this phenomenon could arise in a variety of other model systems with sufficiently singular potentials, and that is one line of discussion we wish to pursue. Before that analysis, however, there are further worthwhile insights to be drawn from our simple example.

Suppose $\alpha \leq 2$ and we initially adopt a rather direct meaning of the singularity of $|x|^{-\alpha}$ equal or equivalent to the "regularization" $V_\epsilon(x) = (|x| + \epsilon)^{-\alpha}$ as $\epsilon \to 0^+$. For each $\epsilon > 0$ the sum $H_0 + \lambda V_\epsilon$ is well defined and one can study the double limit of $\epsilon \to 0^+$ followed by $\lambda \to 0^+$. Whenever $\alpha < 1$ this procedure leads to the free Hamiltonian (harmonic oscillator) and is entirely acceptable in that realm; for $\alpha \geq 1$ this prescription leads to the pseudofree Hamiltonian described earlier as being unavoidable when $\alpha > 2$. To circumvent that outcome in the range $1 \leq \alpha \leq 2$ requires carefully chosen alternative regularizations in order to ensure that the interaction corresponds to a continuous perturbation of the harmonic oscillator. Characteristically, one must choose a regularization such as

$$V_\epsilon(x) = (|x| + \epsilon)^{-\alpha} - \left[\sum_{j=1}^{[(2-\alpha)^{-1}]} k_j \lambda^{j-1} \epsilon^{(2-\alpha)j-1}\right]\delta(x) \quad (3)$$

where the numerical coefficients k_j are determined recursively from $k_1 = 2/(\alpha - 1)$ and

$$k_j \equiv -\frac{1}{[1-(2-\alpha)j]} \sum_{n=1}^{j-1} k_n k_{j-n} \qquad j = 2, 3, \ldots \quad (4)$$

These formulas hold whenever $(2 - \alpha)^{-1}$ is nonintegral; if $(2 - \alpha)^{-1} \equiv J$ is integral the last term in the square brackets in Eq. 3 is replaced by $\bar{k}_J \lambda^{J-1} \ln\epsilon$, where $\bar{k}_1 = -2$ for $J = 1$, and $\bar{k}_J$ is given as in Eq. 4 for $J \geq 2$, apart from the prefactor $-1/[1 - (2 - \alpha)J]$.

Qualitatively, an expression such as Eq. 3 is derived (*3*) by exploiting the fact that near a singularity of the potential any nonvanishing continuous solution $\psi(x)$ of the Schrödinger equation has a dominant and characteristic x dependence, which in virtue of the continuity leads to a definition of the regularized potential as $\frac{1}{2}\psi''(x)/\psi(x)$. Note that the exponent of ϵ in each term in Eq. 3 is negative and its variation with α and j leads to quite specific relative rates of divergence of the coefficients of $\delta(x)$. All such regularizations are arbitrary up to an additional coefficient of $\delta(x)$ that is independent of ϵ, a fact already exploited in deriving the coefficients $\bar{k}_J$. Any coefficient of $\delta(x)$ that vanishes as ϵ vanishes plays no role in the limiting Hamiltonian.

If $\alpha = 2$ all regularizing terms in Eq. 3 diverge as ϵ^{-1} and an infinite number of such terms arise. In this case the regularized potential admits a closed-form expression given by (*3*)

$$V_\epsilon(x) = (|x| + \epsilon)^{-2} - 4\epsilon^{-1}[1 + (1 + 8\lambda)^{\frac{1}{2}}]^{-1}\delta(x) \quad (5)$$

The coefficient of $\delta(x)$ here is equal to that given by Eqs. 3 and 4 for $\lambda < 1/8$ and can be extended to larger values by analyticity or by some other extension technique. But it is significant that while Eqs. 3 and 4 define an acceptable regularization for all $\lambda \geq 0$ and any $\alpha < 2$, the expression in Eq. 5 is unacceptable if $\lambda \geq 3/8$ since a corresponding analytic extension of the eigenfunctions to values of $\lambda \geq 3/8$ leads in the limit $\epsilon \to 0^+$ to nonnormalizable states. As a consequence, suitably augmented perturbation techniques suffice for all $\lambda \geq 0$ when $\alpha < 2$, and for $3/8 > \lambda \geq 0$ when $\alpha = 2$, but they lead to entirely fallacious results when $\lambda \geq 3/8$. This means for $\alpha = 2$ that an alternative solution applies when $\lambda \geq 3/8$, but such behavior is still compatible with the emergence of the free Hamiltonian as $\lambda \to 0^+$.

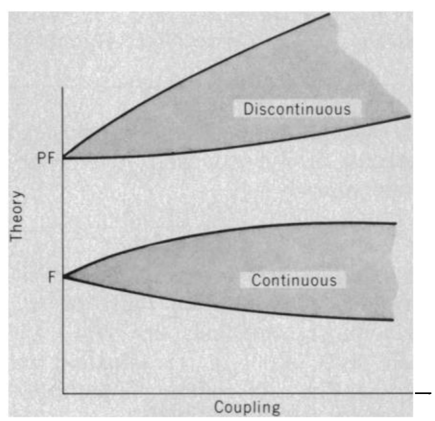

Fig. 1. Highly schematic plot of theory against coupling constant. Here, theory stands for the set of eigenvalues and eigenfunctions, or some other characterizing information such as the collection of Green's functions. Any unperturbed system starts at the point *F* (for free). As the coupling constant of a continuous perturbation increases from zero, the system moves to the right, following a curve in the branch marked "continuous," the specific curve depending on the particular perturbation. When the coupling decreases toward zero any such perturbation passes continuously back to the point *F*. On the other hand, on the introduction of a discontinuous perturbation the system instantly jumps from the point *F* to a new branch marked "discontinuous" and follows a line in that branch which is determined by details of the specific perturbation. Now when the coupling decreases toward zero any such perturbation passes continuously to a new point *PF* (for pseudofree) distinct from the point *F*. Reintroduction of the original perturbation leads to the behavior of a continuous perturbation, not about the point *F*, but about the point *PF*.

Whenever $3/2 \leq \alpha \leq 2$ (and so there are two or more terms in the sum in Eq. 3) it is noteworthy that a λ-dependent regularization is required in order to fashion a continuous perturbation of the harmonic oscillator. It is formally correct to equate the need for λ-dependent counterterms in the anharmonic oscillator example with a corresponding need for coupling-dependent counterterms in the context of quantum field models [for example, the need for a mass counterterm quadratic in the coupling constant in a $(\varphi^4)_3$ theory; that is, a quartic self-coupled covariant scalar field theory in three space-time dimensions]. Indeed the relation of the anharmonic oscillator to quantum field theory is rather like the relation of single-celled organisms to vast and complex higher life-forms in that essential elements of the complex forms are still recognizable in the simpler forms. As α increases toward 2, and the required polynomial in λ enlarges, we relate this situation to increasingly singular superrenormalizable models; when $\alpha = 2$ and an infinite-order polynomial in λ is required, we relate this situation to renormalizable models; and when $\alpha > 2$ and no regularization exists that yields a continuous perturbation of the free theory, we relate this situation to nonrenormalizable models. But for $\alpha > 2$ the anharmonic oscillator has a perfectly acceptable solution when regarded as a discontinuous perturbation of the free theory. Perhaps a similar situation also applies to nonrenormalizable quantum field theories.

Path Space Viewpoint

A broader perspective on discontinuous perturbations may be won in the general framework of a sum over histories; that is, a path space quantization (*4*). Formally, a transition amplitude is expressible in the form

$$A_0 = \sum_{\text{histories}} e^{iI_{0H}} \quad (6)$$

where I_{0H} denotes the classical action I_0 evaluated for the history, H, summed over all histories that respect the initial and final boundary conditions. When a perturbation is introduced the transition amplitude becomes

$$A = \sum_{\text{histories}} e^{iI_H} = \sum_{\text{histories}} e^{i(I_{0H} + \lambda I_{1H})} \quad (7)$$

where I_{1H} denotes the action of the perturbation and λ is the coupling constant. Suppose that the presence of I_1 serves to delete certain histories from the sum so that

$$A = \sum_{\text{histories}}{}' e^{i(I_{oH} + \lambda I_{1H})} \qquad (8)$$

where the prime symbolizes the appropriate deletion. How can such a deletion arise? In simplest terms, a history H will be deleted if $I_{1H} = \infty$ while $I_{0H} < \infty$; that is, if the history is "allowed" by the free action I_0 and "forbidden" by the perturbing action I_1 because of an infinitely rapid oscillation. If one deals with imaginary-time quantum theory, then the summand in Eqs. 7 and 8 is replaced by $\exp[-(W_{0H} + \lambda W_{1H})]$, where W denotes the imaginary-time action; here the conditions $W_{1H} = \infty$ and $W_{0H} < \infty$ delete a history because of an infinite suppression rather than an infinite oscillation. Although oversimplified, such heuristic criteria are nevertheless quite useful and we shall rely on them heavily. The oversimplification has to do with the measure of history sets for which $I_1 = \infty$ (or $W_1 = \infty$) and, more importantly, with the fact that the finiteness of $I_0 + \lambda I_1$ (or $W_0 + \lambda W_1$) is not the real criterion for the support of the histories involved. Occasionally enough is known about the path behavior to give precise criteria, but even then the essence is unchanged. In either the real- or the imaginary-time form, we see that a discontinuous perturbation acts in history space partly as a "hard core" in relation to histories otherwise allowed by the unperturbed action (5).

When significant histories are deleted it readily follows as $\lambda \to 0^+$ that

$$\sum_{\text{histories}}{}' e^{i(I_{oH} + \lambda I_{1H})} \to \sum_{\text{histories}}{}' e^{iI_{oH}} \equiv A_0' \neq A_0 \qquad (9)$$

In other words, as $\lambda \to 0^+$ the transition amplitude is not continuously connected to the free theory but to an alternative, pseudofree theory. Here, expressed in path space language, is an evident manifestation of the indelible imprint left by the perturbation after its removal. Of course, any meaningful perturbation analysis of the interacting theory must be based on the pseudofree system and not the free system.

Relation to conventional hard cores. Situations entirely analogous to the foregoing arise for conventional hard-core problems. However, in the present case it is significant that histories and not configurations are generally relevant. Consider the anharmonic oscillator with $I_0 = \frac{1}{2}\int[\dot{x}^2(t) - x^2(t)]dt$ and $I_1 = \int|x(t)|^{-\alpha}dt$

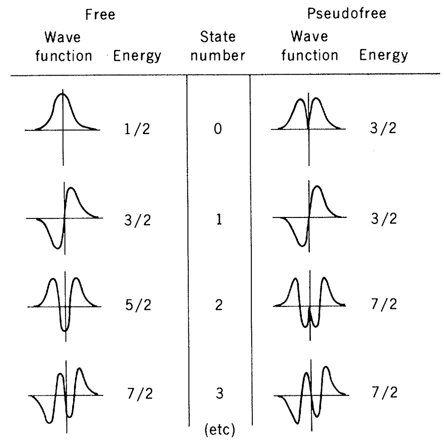

Fig. 2. First four eigenfunctions and eigenvalues for a free harmonic oscillator H_0 and for a pseudofree oscillator H_0' determined as the zero-coupling limit for a perturbation $|x|^{-\alpha}$ for any $\alpha > 2$. Each pair of higher-order eigenfunctions and eigenvalues follows a similar pattern of change.

for the special range $0 < \alpha < 1$. For configurations one requires finite energy, and attention is thus fixed on continuous paths for which $|x(t)|^{-\alpha} < \infty$, and so $x(t) > 0$ (or < 0) for all $\lambda > 0$; as $\lambda \to 0^+$ such restricted paths can never yield all the configurations suitable to the free theory. However, in the sum over histories almost all paths allowed by I_0 satisfy $\int|x|^{-\alpha}dt < \infty$ over any finite integration range, and as a consequence I_1 is a continuous perturbation of I_0.

Necessary condition for a discontinuous perturbation. An important clue to a possible discontinuous perturbation is the presence of histories H for which $I_{0H} < \infty$ and $I_{1H} = \infty$ (or $W_{0H} < \infty$ and $W_{1H} = \infty$). Generally speaking, depending on the model under examination, finiteness of I_0 may imply, for example, that histories are continuous or that they admit logarithmic or power-law singularities. Coupled with each possibility for finite I_0 histories are choices for I_1 that conceivably correspond to discontinuous perturbations. Specifically, negative field powers are required for continuous histories, exponential functions suffice for histories with logarithmic singularities, and suitable positive powers apply for histories with power-law singularities. The relation of the roughness of histories allowed by I_0 and the singularity sensitivity of I_1 for the potentially continuous or discontinuous nature of the perturbation is qualitatively depicted by the nomogram in Fig. 3.

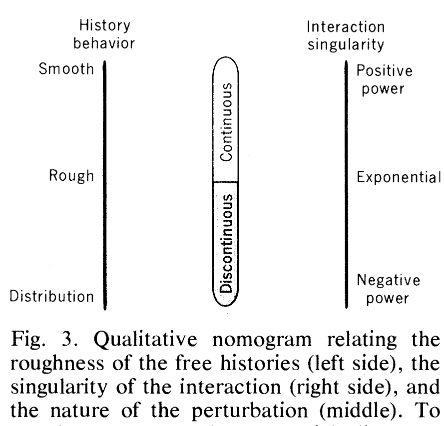

Fig. 3. Qualitative nomogram relating the roughness of the free histories (left side), the singularity of the interaction (right side), and the nature of the perturbation (middle). To use the nomogram, draw a straight line connecting the type of allowed histories (for example, rough) with the type of singularity of the interaction (positive power); the line passes through the middle region and identifies the nature of the perturbation (continuous). Lines that pass through the middle, for a given allowed history type, indicate the minimally singular interaction likely to lead to a discontinuous perturbation.

Interactions that act partly as a hard core in a sum over histories provide a particularly simple picture of discontinuous perturbations. This viewpoint covers anharmonic oscillators—and also, we believe, nontrivial nonrenormalizable field theories. Yet, conventional prejudices against nonrenormalizable field theories are presently so strong that an indirect approach to their discussion is highly advisable. This task we now undertake.

Noise Theory as a Source of Models (Field Theory in Disguise)

A class of simple examples can be given to illustrate the range of possibilities inherent in Fig. 3; moreover, these examples are ultimately relevant to an understanding of field theory as well (6). This class involves generalized types of histories $x(t)$ that may not be pointwise defined and thus are more appropriate to noise theory than to conventional particle mechanics. Choose the imaginary-time form for the action, let $\tilde{x}(\omega)$ denote the generalized Fourier transform of $x(t)$, and take

$$W_0 = \tfrac{1}{2}\int(|\omega|^{2\xi} + 1)|\tilde{x}(\omega)|^2 d\omega \qquad (10)$$

where ξ, $0 \leq \xi \leq 1$, is a parameter at our disposal. Such a choice leads to zero mean Gaussian noise with a power spectrum $(|\omega|^{2\xi} + 1)^{-1}$, which is integrable for $\xi > \frac{1}{2}$ and nonintegrable for $\xi \leq \frac{1}{2}$. The essential nature of the histories varies as ξ varies: smoother histories are associated with larger ξ values; rougher histories are associated with smaller ξ values. Standard imaginary-time quantum mechanics arises for $\xi = 1$, and the associated histories are continuous. In fact, the histories are continuous whenever $\xi > \frac{1}{2}$, specifically fulfilling the condition

$$|x(t) - x(t')| \leq C|t - t'|^{\xi - \frac{1}{2} - \epsilon} \qquad (11)$$

with probability one for any $\epsilon > 0$ and some constant C (and failing with probability one if $\epsilon < 0$). This condition makes it clear that as ξ decreases toward $\frac{1}{2}$ the paths become rougher, although all are continuous. This ξ dependence in allowed path behavior for W_0 reflects itself in a ξ dependence of the class of W_1 expressions that lead to discontinuous perturbations. If $W_1 = \int |x|^{-\alpha} dt$, then divergences arise whenever $\alpha(\xi - \frac{1}{2}) > 1$, that is, $\alpha > 2/(2\xi - 1)$, which is the generalized criterion that leads to discontinuous perturbations (analogous to $\alpha > 2$ if $\xi = 1$).

When $\xi \leq \frac{1}{2}$ the paths are no longer continuous, infinite (but integrable) values are allowed, and singularities of W_1 are won by exploiting the infinite path values rather than any specific finite value, say zero, as in dealing with negative powers. Characteristic singular path behavior is represented by the (distributional) nature of the Fourier transform of the power spectrum, which for small times is proportional to $\ln|t|$ for $\xi = \frac{1}{2}$, to $|t|^{-(1-2\xi)}$ for $\frac{1}{2} > \xi > 0$, and to $\delta(t)$ for $\xi = 0$. Again, the increasing roughness of the paths for decreasing ξ is evident. In the range $\frac{1}{2} > \xi > 0$, pth-order local powers (constructed just like Wick powers) of the path, and denoted by $:x^p(t):$ exist as distributions provided $p < 1/(1 - 2\xi)$. This means that $W_1 = \int :x^p(t): dt$, p even, is finite with probability one for a finite integration range. For $p \geq 1/(1 - 2\xi)$ this is no longer true, but standard renormalization tricks formally suffice to remove divergences whenever $p \leq 2/(1 - 2\xi)$. The construction of these renormalized models is such as to lead to continuous perturbations. If $p > 2/(1 - 2\xi)$ we are in what presently constitutes a no-man's-land of models for which standard renormalization techniques fail, and which thus qualify as "nonrenormalizable" in a sense completely analogous to that of the term in quantum field theory. All this structure can be won just by varying the parameter ξ (which controls the roughness of paths in W_0) and the parameter p (which controls the singularity sensitivity in W_1).

The preceding examples for W_0 and W_1 serve to illustrate that behavior familiar in nonrenormalizable field theories can arise in fairly intuitive stochastic processes. It is noteworthy that the heuristic criterion $W_0 = \frac{1}{2}\int(|\omega|^{2\xi} + 1)|\tilde{x}(\omega)|^2 d\omega < \infty$ implies $W_1 = \int |x(t)|^p dt < \infty$ whenever $p \leq 2/(1 - 2\xi)$, but this implication is false whenever $p > 2/(1 - 2\xi)$. This observation suggests a hard-core interpretation to deal with the so-called nonrenormalizable cases where $p > 2/(1 - 2\xi)$. Concomitant with that interpretation is the view that W_1 represents a discontinuous perturbation of W_0, and that what really is relevant to the analysis is not the free theory but a pseudofree theory.

Shot noise as a nonrenormalizable theory. Evidently the worst case from the point of view of the preceding discussion is the case $\xi = 0$ since the paths are then the roughest and the sensitivity of the interaction term is then the greatest. If this most singular case can be understood it should offer some hope for the other nonrenormalizable cases that are even less singular.

For $\xi = 0$, $W_0 = \int x^2(t) dt$ and $W_1 = \int |x(t)|^p dt$. Let us set our goal on evaluating the path integral

$$C\{s\} = \mathcal{N} \int \exp\{i \int s(t)x(t)dt - \int x^2(t)dt - \lambda \int |x(t)|^p dt\} \mathcal{D}x \qquad (12)$$

where $\mathcal{N}$ is a normalization chosen so that when $s = 0$, $C\{0\} = 1$, and $\mathcal{D}x$ is the differential volume in path space. This integral, as usual, has considerable heuristic but only formal significance. In particular, note that if the term $\int x^2(t)dt$ were replaced by $\int \dot{x}^2(t)dt$ then Eq. 12 would deal with a quantum mechanical problem and would involve continuous paths for which $|x(t)|^p$ is well defined; in its present form Eq. 12 deals with paths not pointwise defined and the meaning of $|x(t)|^p$ as well as $x^2(t)$ is not at all obvious. One thing, however, does seem certain, namely that the answer must have the general form

$$C\{s\} = \exp\{-\int dt L[s(t)]\} \qquad (13)$$

for some real, even function $L[s]$ subject to the normalization $L[0] = 0$. It is clearly the intent of Eq. 12 to exhibit no correlations for unequal times and this is just the content of Eq. 13. A simple argument next shows that the most general such L is given as

$$L[s] = as^2 + \int [1 - \cos(us)] d\sigma(u) \qquad (14)$$

and represents possible Gaussian (a) and Poisson (σ) contributions. The free theory is given by $a = 1/4$ and $\sigma \equiv 0$, and is realized as white noise. On the other hand, every interacting theory involves $a \equiv 0$ and $\sigma \neq 0$, and is realized as (various forms of) shot noise. White noise may be described by

$$x_{\rm WN}(t) = \Sigma\, a_n h_n(t) \qquad (15)$$

where the a_n are independent, identically distributed normal variables and the $h_n(t)$ are elements of a complete orthonormal set of functions. Shot noise may be described as

$$x_{\rm SN}(t) = \Sigma\, u_n \delta(t - t_n) \qquad (16)$$

where the u_n are independent, identically distributed variables (as determined by σ) and the t_n are distributed according to a Poisson law. The two types of noise are fundamentally different, and that fact is reflected in the construction of local powers of the noise. For white noise no prescription proves satisfactory (even allowing for arbitrary renormalization tricks); for shot noise it follows that

$$x_{\rm SN}^p(t) \equiv \lim_{\epsilon \to 0^+} \epsilon^{-1} \left(\int_t^{t+\epsilon} x_{\rm SN}(t')dt' \right)^p = \Sigma\, u_n^p \delta(t - t_n) \qquad (17)$$

describes a perfectly acceptable (renormalized) local power. What is the moral? If local products are relevant—and they certainly are for interactions—shot noise and not white noise is basic.

Fairly straightforward arguments thereafter finally lead to the solution of Eq. 12 being based on

$$L[s] = \int [1 - \cos(us)] e^{-u^2 - \lambda |u|^p} du/|u| \qquad (18)$$

which has the immediate property as $\lambda \to 0^+$ that it does not pass to the free theory (that is, white noise) but instead it passes to a special, pseudofree theory still based on shot noise. In the context of our earlier discussion, every nonquadratic interaction represents a discontinuous perturbation of the free theory, but it represents a continuous perturbation of the pseudofree theory defined by

$$L_{\rm PF}[s] = \int [1 - \cos(us)] e^{-u^2} du/|u| \qquad (19)$$

Note that a meaningful, asymptotic perturbation expansion of Eq. 18 in λ exists relative to the pseudofree theory, but there is no relation whatsoever of these expressions to the free theory.

Clearly, the present example illustrates a hard-core behavior much more complex than that encountered in the anharmonic oscillator, but conceptually the two cases are formally the same.

Relevance and Relation to Field Theory

Traditional studies of nonrenormalizable field theories assume that the principal problem lies in the fact that the dependence of various quantities (such as Green's functions) on the coupling constant λ does not admit a power series expansion about zero (*7*). This behavior is suggested by readily constructed examples in quantum mechanics of perturbations that are continuous but not differentiable at zero coupling. Our proposal is conceptually rather different and asserts that nonrenormalizable field theories generally correspond to discontinuous perturbations. In addition, it is possible that a meaningful perturbation theory does, in fact, exist when related to the appropriate pseudofree theory. What are the grounds for such a view?

Hard cores in field theory. In the imaginary-time formulation of a covariant scalar field $\Phi(x)$ with mass m in n space-time dimensions

$$W_0 = \tfrac{1}{2} \int \{[\nabla\Phi(x)]^2 + m^2\Phi^2(x)\} d^n x$$

$$W_1 = \int |\Phi(x)|^p d^n x \qquad (20)$$

In constructing and evaluating a sum over histories, these expressions enter as free and interacting actions just as their elementary counterparts do in the anharmonic oscillator and noise theory models. Whether W_1 may possibly act as a discontinuous perturbation of W_0 depends on the parameters p and n. Now, Sobolev-type arguments show that $W_0 < \infty$ implies $W_1 < \infty$ provided $p \leq 2n/(n-2)$, but not otherwise. Based on the examples discussed previously, we would expect W_1 to be a continuous perturbation if $p \leq 2n/(n-2)$; but if $p > 2n/(n-2)$, we would expect W_1 to be a discontinuous perturbation. Reference to any standard field theory text shows that this division is exactly the same as that between renormalizable $[p \leq 2n/(n-2)]$ and nonrenormalizable $[p > 2n/(n-2)]$ theories according to conventional renormalized perturbation theory.

It may be easier to accept discontinuous perturbations in the noise theory examples of the previous section than in field theory, a subject in which standard views are deeply ingrained. Yet mathematically the two theories are really quite similar, and structurally they are almost identical. To see this structural identity choose $\xi = 1/n$ and equate the time coordinate of the noise theory with the nth power of the Euclidean distance in the field theory, dispensing with all the angle variables therein except that which allows for a change of the sign of time. Perturbation analyses of the two theories can be developed in parallel, and convergence or (relative) divergence of corresponding graphs leads to an overall equivalent mathematical treatment. To say that a certain form of shot noise, and

"Well, I don't see any point in looking any farther. It was probably just one of those wild rumors."

Fig. 4. Drawing by Chas. Addams; © 1952 The New Yorker Magazine, Inc.

not just white noise or filtered white noise, is relevant as a zero-coupling limit in the noise models inescapably leads one to say that certain non-Gaussian fields are relevant as zero-coupling limits in field theory. Of course, such pseudofree quantum field models need not be in conflict with any general principles; for example, asymptotic fields for pseudofree models would be free fields and would most likely have trivial scattering.

Discontinuous perturbations in quantum theory are potentially as relevant as the more familiar continuous perturbations. They deserve study and analysis with an eye toward possible applications in the real world. As we have related, highly suggestive arguments can be put forth that discontinuous perturbations are relevant for nonrenormalizable field thories. It is certainly tempting to believe them, for in one stroke this would explain the grossly unsatisfactory results obtained through conventional perturbation theory as well as provide a suggestion for determining a meaningful solution. Soluble models show that this is the case and make the goal worth pursuing. Techniques need to be devised to discover and recognize the indelible imprints that discontinuous perturbations invariably leave behind—and one hopes for better insight than displayed in Fig. 4!

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Starvation Kinetics

Bond-breaking denied adequate activation by collision can be starved in many ways.

Henry Eyring

Summary. Reactions that take place in shock waves are slow compared to the energy present in the translational degrees of freedom. One explanation for the slow burning in a detonation is that successive surface layers of the solid or liquid must react sequentially. Another mechanism, which can also account for the slowness of reactions in shock waves in gases, is the basis for starvation kinetics: the bond that is breaking must draw its activation energy from a vibrational reservoir in disequilibrium with the translational degrees of freedom of the reacting molecule.

An interesting place to look for fast reactions is in detonations. Shock waves accompanying detonations in solids move through the explosive at some 40 times the 700 miles an hour of sound in air. The shock initiates the decomposition of the explosive, but the rate of decomposition in solids and liquids is slow considering the temperature of the burning explosive. This is sometimes because layers have to be peeled off sequentially from the surface of the burning particles so that the measured rate is the rate of reaction of a molecule in the surface layer divided by the number of molecular layers lying between the surface and the center of the particle. In a liquid, since the reaction starts from the surface of hot gas bubbles and again progresses layer after layer, the rate per molecule must again be divided by the number of layers to be burned through to obtain the measured rate.

Since this same slowness of reaction is reported in shock waves in the gas phase, another mechanism would have to be active in such cases, and if it is, it would also be expected to appear in some solid explosives. Unimolecular reactions become bimolecular when there are not enough activating collisions to keep decomposition at its high-pressure rate (*1*). This is conveniently thought of as one form of starvation kinetics. The bond that is breaking is no longer fed fast enough to be in equilibrium with the translational degrees of freedom around it, but must draw its activation energy from a starving vibrational reservoir with which it equilibrates. However, there are other ways of starving the reservoir besides simply reducing the pressure of molecules colliding with it. One such way is to introduce inefficient transmissions of energy to the reservoir. This inefficiency varies with the type of coupling made in collisions. An obvious way to produce a starvation process is to use a shock wave. In this case the time or intensity of the shock, or both, can produce a starving reservoir in equilibrium with the hidden breaking bond and lead to a slow reaction rate. That such starvation happens must be obvious to every student of the initiation or dying out of detonations in both gases and solids. In fact, starvation kinetics is the name of the game. This process of starvation of shock waves was discussed at some length in my Priestley lecture (*2*).

Starved reaction reservoirs are conveniently made by dumping measured amounts of energy into the reservoirs by all kinds of means. The formation of various degrees of starved reservoir reactions in mass spectra can be devised by regulating the voltage of the ionizing

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