The Four-Color Conjecture: A Computer-Aided Proof

The four-color conjecture, which posed one of the most famous and, many believe, one of the most difficult problems in mathematics, has now been proved true. Since this conjecture is so well known and since it is easy to state, countless amateur and professional mathematicians have devoted large portions of their lives to attempts at proving or disproving it, and countless erroneous "proofs" have been advanced. The most recent proof, however, differs from its predecessors in that it is accepted by experts in the field of combinatorics, who are intimately involved with ramifications of the four-color conjecture. Moreover, these experts include some of the greatest skeptics of previously acclaimed "proofs."

The four-color conjecture can be most simply stated as a problem in map coloring. A map consists of a division of a plane into regions. For example, a map of a state divided into counties is such a division. Coloring a map consists of assigning colors to the regions of the map. According to the conjecture, the regions of any map on the plane can be colored with four or fewer colors so that no two adjacent regions are assigned the same color. Many maps have been devised that cannot be colored in this way with fewer than four colors so mathematicians knew that at least four colors are necessary to color any arbitrary map. To resolve the four-color conjecture, then, it was necessary either to show that every map that can be drawn on the plane can be colored with four colors or to find a counterexample. Recently Jean Mayer of Université Paul Valéry in Montpellier, France, proved that if any map existed that could not be colored with fewer than five colors, it would have to have at least 96 regions. Thus mathematicians knew that if the conjecture was false, it would be extremely difficult to find a counterexample.

The method finally used to prove the four-color conjecture is an old method—it was first suggested by A. B. Kempe in 1879—that turned out to be extremely difficult to implement. The proof was carried out by Kenneth Appel and Wolfgang Haken of the University of Illinois. According to Appel, their method of proof had never before worked for the four-color conjecture because so many maps needed to be analyzed that a computer was required to carry out the necessary calculations. Appel estimates that the proof required about 10 billion logical decisions and Haken reports that the calculations would take about 300 hours on a very fast existing computer. (Appel and Haken did not use such a computer for all their calculations, so their calculations actually took about 1200 hours of computer time.)

Appel and Haken followed the usual mathematical convention of converting maps to graphs in order to attack the four-color conjecture. (A graph is a set of points connected by lines.) Every map can be associated with a unique graph by identifying each region with a vertex and connecting two vertices of the graph if their corresponding regions of the map have a common boundary (Fig. 1). The four-color conjecture for maps then translates into the statement that four colors are sufficient to color the vertices of any graph on the plane, so that no two vertices connected by a line are assigned the same color.

The essence of the proof by Appel and Haken is as follows. They found a collection totaling 1936 graphs with the property that any graph drawn on the plane must contain at least one of the 1936 graphs within it and that each of the 1936 graphs is what is known as a reducible configuration. It had been previously shown that if every graph contains a reducible configuration, then the four-color conjecture is true. Therefore, since Appel and Haken showed that every possible graph on the plane contains one of their set of reducible configurations, they showed that every possible graph can be colored with at most four colors.

Appel and Haken used a computer to find a means to specify, by hand, the set of 1936 graphs that are contained in all other graphs and they and their colleague John Koch used a computer to prove that each of the 1936 graphs is a reducible configuration. Proving that a particular graph is reducible can be a time-consuming process. Appel, Haken, and Koch limited themselves to a half hour of computer time for each proof of reducibility. If a particular graph had not been shown to be a reducible configuration in that time, they abandoned it and went on to another graph. They point out that the 1936 graphs which they eventually came up with are not the only set of reducible configurations that are contained within all other graphs. It remains possible that a smaller set of reducible configurations will suffice to prove the four-color conjecture, but Appel and others believe that the minimum number of reducible configurations will not turn out to be so much smaller than 1936 that a computer would turn out to be unnecessary to their proof.

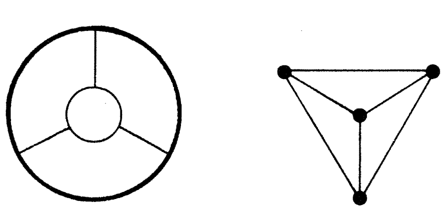

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Fig. 1. (A) A map that cannot be colored with fewer than four colors so that no two adjacent regions are assigned the same color. (B) The graph corresponding to (A).

Now that the four-color conjecture has been proved true, what lies ahead for those who have devoted their lives to this problem and problems that grew out of it? The predominant response among mathematicians is that it is unlikely that the resolution of this conjecture will change the character of research. Some famous unsolved problems, such as the Riemann hypothesis, have huge bodies of work based on the supposition that they are true. Thus if the Riemann hypothesis were proved either true or false, the status of important mathematical theories would be altered. This is not the case, however, for the four-color conjecture. Instead, according to Ronald Graham of Bell Laboratories in Murray Hill, New Jersey, the basic benefit of the four-color conjecture has already been realized. Attempts to prove it have led to tremendous developments in the field of combinatorics and have provided a major stimulus to the growth of graph theory. Now these spin-offs from the four-color conjecture have taken on lives of their own, independent of their source.

William Tutte of the University of Waterloo in Canada is an example of a mathematician who has seriously studied various aspects of the four-color conjecture and who expects life to go on as usual now that it is proved. Tutte, who was skeptical of and found an error in a previous well-publicized "proof" of the four-color conjecture, believes that Appel and Haken have really proved this conjecture. His own work, however, has branched off into the question of finding the average number of colors sufficient to color various collections of graphs. This problem, although tied to the four-color conjecture, is not solved by the resolution of the four-color conjecture.

Since the proof by Appel and Haken required that so many logical decisions be executed by a computer, it raises the question of whether there is some simpler way to obtain the result and, if not, whether the computer proof can be trusted. Such dilemmas arising from the increasing likelihood that computers will be used to yield otherwise unobtainable results in mathematics are a subject of heated debate among mathematicians and computer scientists (*Science*, 4 June 1976, p. 989). Some graph theorists believe that their colleagues will continue to search for a proof of the four-color conjecture that does not rely on a computer. On the other hand, even those who believe in the method used by Appel and Haken are bound to check their calculations. In fact, Frank Allaire of the University of Manitoba in Canada and Edward Swart of the University of Rhodesia are attempting to prove the four-color conjecture by a method analogous to that of Appel and Haken but with different computer algorithms.

The proof of the four-color conjecture, then, does not put an end to anything in mathematics except perhaps the speculation that this problem could not be solved. Research on problems derived from the four-color conjecture continues unchanged, and the search for a simpler proof of this conjecture is not likely to be abandoned. Some mathematicians, though, believe that the requisite use of a computer to solve this problem may herald a new era in mathematics research in which human theoreticians find themselves increasingly relying on computers for unavoidable calculations that will prove necessary to solve difficult problems.—GINA BARI KOLATA

Computers: Helping to Study Nerve Cell Structure

As neurobiologists probe more deeply into the function of the nervous system their need for quantitative data about nerve cell structure increases. They want to know, for example, about the branching patterns of neurons, about the number of connections that one neuron makes with others, and about how these properties are altered by genetic and environmental changes that affect behavior. The idea is to correlate structure with function.

But neurons normally have many branched projections that extend in a three-dimensional network over distances that are large compared to the size of typical cells. Thus, determining complete neuronal structures by standard cytological techniques is both tedious and time-consuming. More important, the structural data so acquired are difficult to store and record in useful form. Recently, however, a number of investigators have developed computer-assisted systems that gather, store, and analyze neuroanatomical data. The systems are helping to put neuroanatomy on a quantitative basis. Moreover, many investigators think that the systems are now producing information that could not otherwise be obtained.

Reconstruction of the three-dimensional structure of neurons is one of the principal uses of the computer systems. Several investigators have developed them for this purpose. They include W. Maxwell Cowan, Donald Wann, and their colleagues at the Washington University School of Medicine, Cyrus Levinthal of Columbia University, with Randall Ware, who is now at the California Institute of Technology, Paul Coleman and William Simon of the University of Rochester Medical Center, Rudolfo Llinas and Dean Hillman of the University of Iowa, and Robert Lindsay of the University of California in Los Angeles.

One way to define the three-dimensional structure of a neuron is to determine the spatial coordinates of its anatomical features. A neuron consists of a cell body, several dendrites, and usually one axon. The axon is a long projection that carries nerve impulses away from the cell body to the target neurons or other target cells. The dendrites, which frequently have a treelike branched structure, transmit incoming impulses from other neurons toward the cell body. Knowing the x, y, and z coordinates of features such as the points of origin of the dendrites on the cell body and their branch and end points would enable the investigator—or the computer—to reconstruct the dendrite structures.

Some systems use computer-assisted microscopes to obtain the coordinates directly from preparations of stained nerve tissue. The stain most often used is the Golgi stain, which is named after Camillo Golgi, who developed the technique in 1873. It involves impregnating neurons, usually in brain or spinal cord, with silver. The silver normally stains only a small fraction of the nerve cells in the tissue, but these stand out clearly against a translucent background. Impregnation of the dendrites is virtually complete, although the axon may or may not be stained. The stained nerve tissue is then sliced into relatively thick (about 100 micrometers) sections and placed on slides for microscopic study (Fig. 1A).

The stage controls of any microscope move the slide in the x and y directions. Adjustment of the focus control determines the depth of the focal plane being examined; this is the z direction. The stage and focus controls of the computer-assisted microscopes are driven by stepping motors, which move the controls in small increments determined by the investigator. The motors are hooked up with the computer in such a way that the movements of the controls can be monitored and their positions recorded when the feature of interest has been brought into sharp focus.

Most of the current systems are considered semiautomatic because they require a human operator to track the neuronal projections by manipulating the motor controls and to signal the computer when to record a set of coordinates for a particular feature. The operator must first select a structure, usually the cell body, to serve as a reference point from which the coordinates are measured.

Many investigators think that fully automatic recognition of patterns as complex as those of neuronal structures are beyond current capabilities to devise programs for the computer. Although Coleman, for example, has designed a system that focuses on and tracks dendrites automatically, it still requires an operator to monitor the process and to make corrections if needed. The operator must also answer the computer's queries about identification of end and branch points. This system reduces but does not eliminate the need for an operator.

However, the computers of even the semiautomatic systems are programmed to "remember" branch point positions and to return to them automatically in order to insure that all the branches are tracked. In other words, computers have good memories but poor pattern discrimi-