Moving Boundary Problems: Advances on Several Fronts

Billions of barrels of oil lie trapped in old oil fields that have been pumped so long they have lost their natural pressure. As the cost of oil shoots up, oil companies are turning with increased interest to the problem of how best to extract this oil. One way is to flood the perimeter of a field with water, thus forcing the oil to some central area where it can be pumped. The injection of water into the field and the rate of pumping must be carefully controlled to maximize the amount of oil that can be extracted before water starts coming out. Because this operation is so expensive, oil companies want to know, as exactly as possible, the rate at which the boundary between the oil and water moves and how the movement of the boundary is related to the amount of water injected into the field and the rate the oil is pumped out. The resulting mathematical problem is an example of what is known as a moving boundary problem.

Moving boundary problems turn up in nearly every field of science and in numerous situations that arise in industry. They occur, for example, in descriptions of the diffusion of gases and the movement of convective boundaries during the evolution of stars, in models of the diffusion of oxygen into tissues, in questions of how to control the welding and drilling of metals with lasers, and in economic models used by business and industry. Most of these moving boundary problems are of great practical importance, but are notoriously difficult to formulate and solve. Investigators have, however, recently devised new ways to tackle several classes of moving boundary problems, with the result that some problems that once seemed intractable now appear much less formidable and some of the simpler problems are so well analyzed that they are said to be no longer of interest to theorists.

The archetypical moving boundary problem, called the Stefan problem, concerns the melting of a piece of ice in a body of water. The temperatures of the regions of water and ice at some initial time are given. The problem is to describe how those temperatures change with time. As the ice melts, the boundary between the two regions moves.

Temperature changes in the ice region can be described by a partial differential equation—the heat equation—and temperature changes in the water region can be described in the same way. Difficulties occur, however, at the boundary, where there are sharp discontinuities in the heat equation. One reason for these discontinuities is that the heat capacity and thermal conductivity of water are different from those of ice, and these parameters show up as coefficients in the heat equation. Another reason is that there is a latent heat associated with the change of phase from water to ice, and this latent heat must be accounted for.

Before choosing numerical approximation schemes to solve moving boundary problems or searching for means to deal with the discontinuities at the boundaries, mathematicians generally want to know whether a solution exists, whether it is the only solution, and how it behaves. For example, they ask, will the boundary be sharp or diffuse, smooth or irregular? John Cannon of the University of Texas at Austin and his associates spent the past 8 years studying such theoretical questions about one-dimensional moving boundary problems; and, according to Cannon, they have recently "finished off" these theoretical questions.

From Cannon's work, it is now known that the boundary of one-dimensional problems, which is a point, will be sharply defined and will move smoothly. On the other hand, solutions to moving boundary problems posed in two or more dimensions may not have such nice properties. Irregular boundaries are to be expected, and the solutions to the equations will be so-called weak solutions, meaning that they are not necessarily smooth enough to serve directly as solutions to the partial differential equations but that they do solve these equations in a more general sense.

Front Tracking Methods

Even one-dimensional moving boundary problems can prove difficult to solve. Several investigators, however, have devised numerical methods, called front tracking methods, that work well for most one-dimensional moving boundary problems.

When using a front tracking method, researchers first calculate where the front, or boundary, will have moved at a given time by approximating a solution to an ordinary differential equation that describes the front's motion. Then, they freeze the front at that position and find an approximate solution to the heat equations for ice and water on either side of that front. Repeating this process at each time step, they can get a good idea of how the front moves and how the temperatures throughout the region change with time.

These front tracking methods are not easily extended to problems involving two or more dimensions. A major difficulty lies in the geometry of the problem. It is much more difficult to store enough information in a computer to specify a curve, which is a two-dimensional boundary, than a point. Moreover, numerical approximation methods used to solve two-dimensional problems, such as the finite element method (*Science*, 27 May 1974), do not converge as quickly when the boundary is irregular. Thus many investigators are seeking and developing alternative means to solve moving boundary problems in two or more dimensions.

Gunter Meyer of the Georgia Institute of Technology and others have obtained numerical solutions to certain two-dimensional problems by artificially smoothing out the discontinuities across the moving boundary and solving the resulting nonlinear partial differential equations on the entire region of the problem. Such methods have the disadvantage that they can involve formidable computations.

An alternative technique, recently developed by Alan Berger, Melvyn Ciment, and Joel Rogers of the Naval Surface Weapons Center in Silver Spring, Maryland, avoids this drawback and turns out to be extremely simple and efficient to implement. To solve the Stefan problem with their new method, Berger, Ciment, and Rogers treat the heat flow in the ice and water in two separate steps at each time interval. In the first step, they add sufficient heat to the ice region to convert it to water and subsequently solve the heat equation for an all-water region. Then, they subtract the heat they had added to the ice. In the next step, they subtract heat from the water region to make it ice, solve the heat equation for an all-ice region, and add the heat they took away. By repeating this process at each time interval, they end up with temperature distributions of the ice and water and boundary positions (the boundary shows up as the region whose temperature is 0°C) at various times (Fig. 1).

Berger, Ciment, and Rogers also used their new method to solve the two-dimensional Stefan problem and a similar problem that models the diffusion of oxygen into a tissue. As the oxygen seeps in, the boundary between the region of tissue with and without oxygen moves. The problem is to describe the motion of this boundary and the concentrations of oxygen in the tissue as a function of time. Since their method worked so well for this problem and for the Stefan problem, Berger, Ciment, and Rogers suggest that similar ideas may provide an easy way to solve other moving boundary problems such as those involving shock waves.

Some very difficult and important prob-

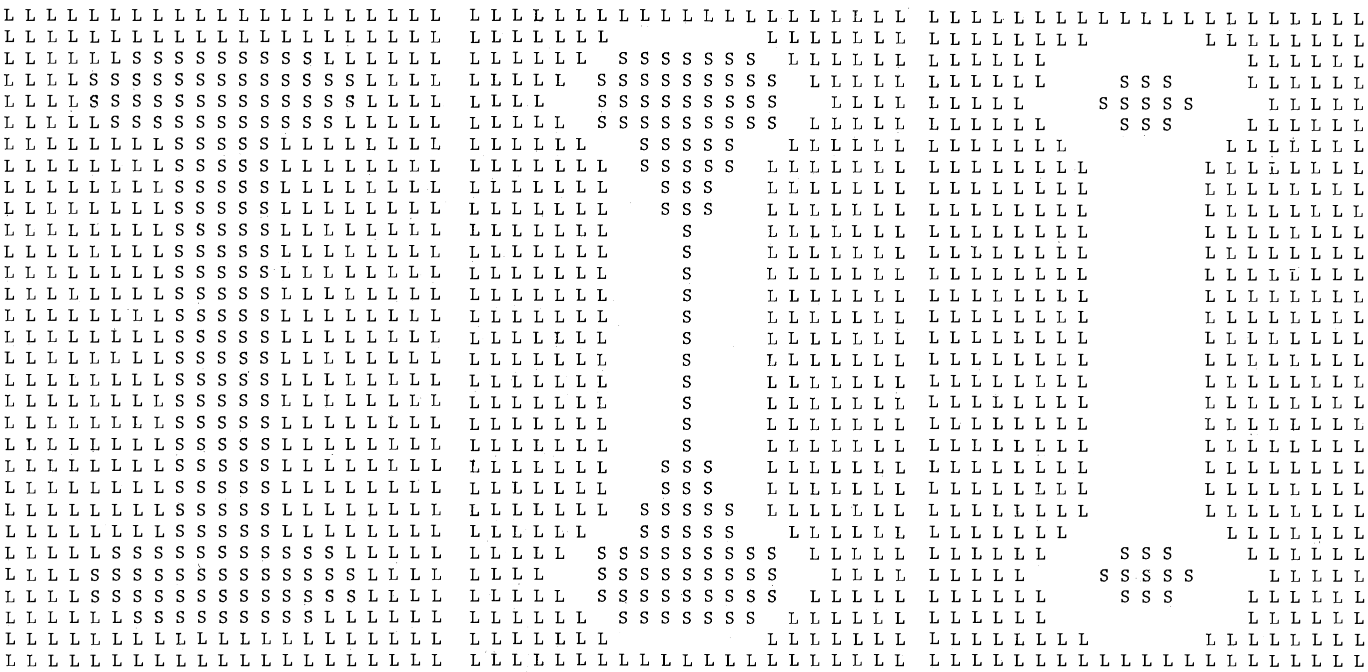

Fig. 1. The dog bone problem. Location of liquid (L), solid (S), and intermediate "slush" region (blank) at several times during melting of a solid "dog bone." Numerical solution obtained by method of Berger, Ciment, and Rogers. (Left) Initial time, dog bone has not yet begun to melt. (Middle) About halfway through computer run. (Right) Near end of computer run.

lems, however, are not equivalent to the Stefan problem, and it is likely that smoothing techniques and techniques such as that devised by Berger, Ciment, and Rogers will not suffice to solve them. Those methods work for problems such as the Stefan problem because the position of the moving boundary is completely determined by the governing partial differential equations in the entire region and the conditions on the boundary of the region. When extraneous conditions affect the moving boundary, the methods of solving the problem will likely be much more complex.

One example of a problem with external conditions imposed on the boundary arises in the cutting of metals. When designing ways to cut metals with high-power lasers, engineers wish to find a means to drill a hole through the metal without allowing the molten metal to reenter the hole and solidify. This is a moving boundary problem in which the molten and resolidifying metal forms the moving boundary. The boundary has an external input, though, and this is the laser radiation used to drill the hole.

Meyer proposes that certain difficult moving boundary problems involving two or more dimensions, and having some sort of external conditions imposed on the boundaries may be solved by turning them into sequences of one-dimensional problems. Front tracking methods may then be applied to these one-dimensional problems. Meyer points out that such transformations of multidimensional problems into sequences of one-dimensional problems have already been reported by Russian investigators, who used this trick to artificially smooth the boundaries of Stefan problems. However, this method has never previously been used to track fronts.

Meyer has developed two distinct locally one-dimensional methods to track fronts of multidimensional problems. One, called the method of alternating directions, is based on a decomposition of the movement of the boundary into its components along the coordinate axes. With this method, if the solution at time t_n is known and the solution at time t_{n+1} is desired, the interval between time t_n and t_{n+1} is partitioned as the front is tracked. Meyer uses this technique to track a two-dimensional boundary in the following way. First, he fixes the y coordinate at time t_n and solves for the new x coordinate of the front at time $t_{n+1/2}$. Then, he uses this solution for x to solve for y at t_{n+1}. In this way, the front is advanced by half steps, and at each half step he solves for only one coordinate, making the method locally one-dimensional. And, at each half step, Meyer uses a well-tested method to solve the one-dimensional equation that describes conditions in the regions on either side of the boundary.

With his second method, Meyer singles out the dominant direction of movement of the boundary. He then approximates the multidimensional heat equation with a system of coupled one-dimensional equations describing the motion of the front in this direction. This system of equations is solved iteratively by the application of a front tracking algorithm to one equation at a time. This algorithm is largely independent of the specific form of the conditions on the moving boundary, which enables Meyer to take into account external inputs, such as heat, on the boundary.

Another approach to solving moving boundary problems is to change the form of the problems so that they become inequalities involving integrals (variational inequalities) rather than partial differential equations. Such restatements of moving boundary problems offer several advantages. One advantage is that mathematicians have built up a large body of theory to deal with variational inequalities, and this work can be utilized in solving the restated moving boundary problems. In addition, when problems are formulated as variational inequalities, the statement of the problem incorporates all the conditions on the regions and the boundary in a single expression and, at the same time, forms a natural context for weak solutions and their approximations.

Several investigators have successfully solved difficult moving boundary problems by posing them as variational inequalities. One proponent of this procedure, Avner Friedman of Northwestern University, believes that his use of variational inequalities has allowed him to obtain results which he could not have otherwise obtained, about the shape and regularity of boundaries of melting ice and of oil seeping through the ground.

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significant, and women who did not have their first pregnancy until their late twenties. Finally, women who took the pill for 2 to 4 years appeared to have a somewhat higher incidence of breast cancer than women who took it for less than 2 or more than 4 years. Paffenbarger said that these data may indicate that contraceptive pills accelerate the development of preexisting but undetected cancers, rather than initiating the development of new cancers.

Other studies have not uncovered any relationship between oral contraceptives and breast cancer. One of the largest of these is a prospective study including 23,000 women that has been conducted since 1968 by the Royal College of General Practitioners in England. This study and several others have also shown that there is a decreased incidence of benign breast disease in pill users. Since benign conditions predispose to breast cancer, this means that the pill may actually be protective for some women.

The Walnut Creek Study

In the United States, the Walnut Creek Contraceptive Drug Study carried out under the auspices of the Kaiser-Premanente Health Foundation in Walnut Creek, California, is another large prospective study concerned with the side effects of oral contraceptives. According to Savitri Ramcharan, the study director, it involves a total of 18,000 women who have participated in the program for up to 7 years. Because of the rarity of cancer in women of reproductive age—probably no more than one case per 1000 women per year—Ramcharan says that they are just now accumulating enough cases to give statistically significant results. It will be several months before an analysis of the data is completed.

Because of the importance of questions concerning the effects of oral contraceptives on cancer incidence, Heinz Berendes of the National Institute of Child Health and Human Development (NICHD), the institute with primary responsibility for research related to oral contraception, wrote to all the investigators under contract to NICHD to determine whether they had collected data that might provide answers. Unfortunately, most investigators did not have appropriate data. Conducting epidemiological studies that are large enough to produce statistically significant results in a reasonable period of time is difficult. Thus, it may be a while before there is a definite answer to questions about the pill and cancer.

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Mathematicians are also using variational inequalities to solve optimal stopping problems, which are stochastic problems derived from economics. These problems, unexpectedly, turn out to be equivalent to moving boundary problems. An optimal stopping problem considered by Friedman and his colleague Robert Anderson of the University of Pittsburgh involves quality control in a manufacturing plant. Two products are made by a machine: product A, which is a good product, and product B, which is made when the machine goes haywire but which cannot easily be distinguished from product A. The director of the plant loses money each time he checks the machine to determine whether it is making product A or B. On the other hand, he also loses money if he does not check and product B is made but not detected. The question, then, is how often should he check the machine so as to minimize his costs? The question is a stochastic one because there can be only probabilistic estimates of how often the machine will go haywire and make product B. Friedman and Anderson showed that such optimal stopping problems can be transformed into variational inequalities. They are solving these inequalities with analytical techniques.

J. L. Lions and Alain Bensoussan of the Institut de Recherche d'Informatique et d'Automatique in France are also solving optimal problems with variational inequalities. For example, these mathematicians analyzed a problem involving the ordering of stock from a warehouse. A business must pay a service charge each time it places an order for stock, independent of the amount of stock ordered. The business must also pay a penalty if it runs out of stock. The question, then, is how often should stock be ordered so as to minimize costs? The problem is a stochastic one because there can be only probabilistic estimates of how much stock will be purchased by customers at any time.

Perhaps because moving boundary problems are of such practical importance, the pace of research on these problems has been extremely rapid. A great deal of work begun in the past decade has now come to fruition and, consequently, the entire face of the field has changed. Now, many believe, there is reason to be optimistic that practical problems involving moving boundary problems can be solved.

—GINA BARI KOLATA

Additional Reading

1. J. R. Ockendon and W. R. Hodgkins, Eds., *Moving Boundary Problems in Heat Flow and Diffusion* (Clarendon, Oxford, 1975).