Cascading Bifurcations: The Mathematics of Chaos

Mathematical models of physical and biological phenomena are often criticized because the phenomena are exceedingly complex, yet the models contain very simple equations. However, recent results in the mathematical field of bifurcation theory indicate that complicated, even random, events may be modeled by simple nonlinear equations. Such equations often have solutions so complex that their behavior appears completely chaotic. Investigators are beginning to develop a general theory to describe the mathematical structure of such chaotic solutions and are using these results to model very complex phenomena.

Bifurcation theory involves the study of equations whose solutions branch, or bifurcate, as a parameter (the bifurcation parameter) in these equations varies. Such equations often occur in models used in mechanics, fluid dynamics, elasticity, and population dynamics. The bifurcations of the solutions to these equations means, physically, that the systems can exist in any of several states. For example, in the nonlinear equations modeling the buckling of a rod that is subjected to increasing pressure, the applied pressure is the bifurcation parameter. When the pressure reaches certain values, the solutions to these model equations branch. The branch points correspond to critical pressures that can cause the rod to buckle in any of a number of ways, depending on which solution branch is followed. Physically, one course will be followed; mathematically, it is indeterminate which occurs.

Since many models of physical events consist of equations whose solutions bifurcate, bifurcation theory is of considerable practical importance. Bifurcations were first described nearly 200 years ago by the Swiss mathematician Leonhard Euler and have been studied ever since. However, until recently, investigators have focused on simple bifurcations that occur when a solution first splits into two or more branches. More complicated behavior can occur when these branches of a solution split again to form secondary bifurcations. The branches from the secondary bifurcation points sometimes split to form tertiary bifurcations, and this process can continue to form a complex tree of solutions that may have infinitely many branches (Fig. 1). As a bifurcation parameter is varied, a solution can follow any path in such a tree or can jump from branch to branch at random.

Several investigators are studying the chaotic behavior of solutions to very simple nonlinear equations used to model biological and physical events. Robert May of Princeton University, for example, analyzed several nonlinear equations that are often used to describe the growth of fish or of insect populations, such as 13-year periodic cicadas, that have nonoverlapping generations. For such populations, the number of organisms alive at a given time is a function of the number alive in the preceding generation. Equations modeling the growth of these populations are known as difference equations.

May reports that whether these nonlinear difference equations have solutions that behave in a simple way depends on the size of a growth rate constant (r), which relates a population's size to the size of the preceding population. As the parameter r increases, the population's size begins to oscillate between two points, then it oscillates among four points, then among eight points, and so on until it eventually enters a chaotic regime. In the chaotic regime, the population's growth occurs in cycles of arbitrary periods or with no periodicity at all, depending on the size of the initial population. According to May, r need not be large for chaos to ensue.

May's results imply that populations whose initial sizes are arbitrarily close can have growth trajectories that differ greatly. Thus even if parameters of an equation modeling a population's growth could be known exactly, the dynamics of the population's growth could be unpredictable. Apparently random fluctuations in sizes of animal populations are sometimes attributed to unknown environmental causes or to sampling errors. However, it remains possible that these random fluctuations are due to growth rates described by nonlinear equations with chaotic solutions.

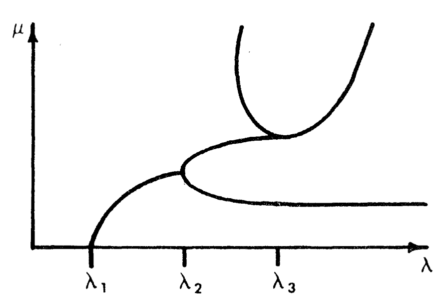

Fig. 1. Bifurcations of solutions to a hypothetical nonlinear equation, $F(\mu, \lambda) = 0$. In this example, $\lambda = 0$ is a solution to the equation. This solution branches at λ_1, a primary bifurcation point. Secondary and tertiary bifurcations occur at λ_2 and λ_3.

Tien-Yien Li and James Yorke of the University of Maryland at College Park analyzed the large class of nonlinear difference equations of the form $N_{t+1} = F(N_t)$, where N_t is the value of the dependent variable N at time t and F is any continuous function. They report that chaotic behavior will result whenever N increases for two time intervals and then decreases to a value less than or equal to its original value (that is, when $N_{t+3} \leq N_t$ and $N_t < N_{t+1} < N_{t+2}$). This situation occurs in various models of population growth and epidemiology, and in other contexts such as models of the distribution of points of impact on a spinning bit used for drilling oil wells. In models of population growth, N_t would denote a population's size at time t. A population may grow in size for two generations whereupon it reaches a size that cannot be sustained and falls back in the third generation to a size less than or equal to its original size. If such a population's growth is accurately described by a nonlinear difference equation, the results of Yorke and Li imply that the population will have chaotic growth dynamics.

Yorke points out that his results indicate that solutions to even the simplest nonlinear difference equations may behave chaotically. Statistical methods are, apparently, necessary to describe the long-term average behavior of the chaotic sequences N_t. In collaboration with Andrezej Lasota of Jagellonian University in Cracow, Li and Yorke have developed a probabilistic theory that can lead to a statistical mechanics for a large class of these functions F. However, a number of questions remain unresolved for even the simplest of these equations.

The unpredictable behavior that results when an event is modeled by a nonlinear differential equation (a differential equation is a continuous analog of a difference equation) that has numerous, or even infinitely many, solution branches has only recently been studied in a general way. Louis Bauer of the Courant Institute of Mathematical Sciences at New York University, Herbert Keller of the California Institute of Technology, and Edward Reiss of the Courant Institute have developed a theory that can be used to determine when higher order bifurcations will occur in the solutions to general nonlinear differential equations.

Bauer, Keller, and Reiss began by finding when solutions to nonlinear equations will have secondary bifurcation points and where those points will occur. Others had shown previously where primary bifurcation points (the initial branch points of a solution) will occur. If a nonlinear equation is approximated by a linear equation, primary bifurcation points of the solution to the original nonlinear equation occur at specific values, the eigenvalues, of the solution to the linearized equation.

Bauer, Keller, and Reiss used numerical and perturbation methods to show that when eigenvalues of the linearized equation are of a certain type (when they are multiple eigenvalues) secondary bifurcations can occur. Reiss reports that tertiary bifurcations can occur when the secondary bifurcation points are multiple eigenvalues of the problem linearized about the primary bifurcation solutions. He extends this analysis to provide the basis of a general theory, which he calls the theory of cascading bifurcations.

Cascading bifurcations have been observed in the buckling of plates and shells. If a plate is subjected to increasing pressure, it may buckle when the pressure reaches a certain critical value. If the pressure is then increased, it reaches other critical values at which the plate may buckle further. In the mathematical model of plate buckling, each critical pressure corresponds to a bifurcation point of the solution to a nonlinear equation. At each critical pressure, the plate can assume any one of a number of buckled states, corresponding to several solution branches at each bifurcation point.

Bernard Matkowsky and Thomas Mahar of Rensselaer Polytechnic Institute are applying Reiss's method of calculating cascading bifurcations to a mathematical model that some believe may describe the phenomenon of morphogenesis. The model is based on a hypothesis advanced by A. M. Turing, then at the University of Manchester, in 1952. Turing proposed that biological forms can arise from a homogeneous mixture at chemical equilibrium when that mixture is perturbed. The perturbation would set off a sequence of instabilities in the chemical mixture. This concept was formalized by Ilya Prigogine of the University of Brussels. Prigogine proposed a hypothetical "trimolecular reaction" that could be used to model this process. The equations representing this group of reactions are assumed to have solutions that undergo a cascade of bifurcations. At each bifurcation point, the reactions of the chemical mixture would become increasingly complex until, finally, the mixture would take on a new and highly ordered form. Investigators had previously demonstrated only primary bifurcations of solutions of these chemical equations. Now, Matkowsky and Mahar report that secondary, and possibly cascading, bifurcations occur in these solutions.

Bauer, Keller, and Reiss hope to use their theory of cascading bifurcations to describe the way a smoothly flowing liquid becomes turbulent as the strength of the flow is increased. How to describe this process has baffled physicists and engineers for decades.

One hypothesis to explain turbulence was advanced in 1941 by Russian physicist D. Landau. Landau suggested that turbulence may be described by equations whose solutions undergo a sequence of bifurcations. At each bifurcation, the fluid flow would become more complex until finally it would become completely chaotic. The increased emphasis among mathematicians on describing cascading bifurcations and determining when they will occur may lead to a determination of whether Landau's hypothesis can be used to describe the transition to turbulence.

The established field of bifurcation theory is now the subject of renewed interest due to the new theoretical results in cascading bifurcations and chaotic regimes. It is certainly possible that many complex phenomena may be modeled and analyzed in terms of simple nonlinear equations whose solutions appear as cascading bifurcations.—GINA BARI KOLATA

Paleontology: Facing a Choice Between Fossils and Trash

The destruction of historic sites seems more and more to be becoming a commonplace occurrence as man pushes aside the past to build for the future. In many cases, the destruction of artifacts may be unavoidable or perhaps even justified; taken to an extreme, the desire to preserve history would require that nothing ever be destroyed. But occasionally there arises a case where it is patently obvious that preservation is the only acceptable alternative. Such is the case with the Messel Pit, a major paleontological site in West Germany that, if current government plans are carried through, may soon be filled with trash and other wastes.

The Messel Pit, located about 10 kilometers northeast of Darmstadt, is the site of a lake that formed at the junction of two fissures during the Eocene epoch, more than 50 million years ago. The lake, according to Jens Lorenz Franzen and Siegfried Rietschel of the Senckenberg Research Institute and Museum of Natural History in Frankfurt, apparently contained very little oxygen. Consequently, flora and fauna that fell into the lake underwent little degradation before they were eventually covered by the accumulating sediment. The end result was the formation of a large deposit of oil shale—stratified rock containing as much as 22 percent organic matter and large numbers of well-preserved fossils.

Most of the oil shale was mined over a long period ending in 1971; it was used for a variety of purposes, including the production of medicinals, chemicals, and gasoline. The mining left a large crater about 1000 meters long, 700 meters wide, and 60 meters deep. The crater still contains many pockets of oil shale, and it is the fossils found in this shale that are a source of international concern.

German paleontologists have long recognized that the Messel Pit is the source of excellent specimens. For many years, the Darmstadt Landesmuseum had exclusive rights to dig at the site; but the museum had insufficient funds to sponsor the work, so it let amateurs use the site without supervision. Consequently, most of the better specimens from the site have ended up in private collections without having been catalogued. Only very recently has a more concerted effort been carried out at the site by teams of investigators from the Landesmuseum and the Senckenberg Institute.

The specimens they have obtained so far are quite exceptional. Whereas digging at other sites most often produces remnants of fossil skeletons or jumbled collections of fossil bones, Rietschel says, excavation at Messel commonly produces complete skeletons in a remarkable state of preservation. Digging in the past few weeks, for example, has yielded an intact skeleton of an eohippus, a small horse of the Eocene epoch; the skeleton of a large bat with two smaller bats attached at the breasts; and the skeleton of a large snake with, inside it, the intact skeleton of a young crocodile. In many cases, Rietschel says, it is even possible to observe the shadow of the soft