The 1974 Fields Medals (II): An Analyst and Number Theorist

Enrico Bombieri was awarded the Fields Medal for his work in both number theory and minimal surfaces. Indeed, Bombieri's unusual versatility extends to still other fields of mathematics. He has repeatedly demonstrated an ability to quickly master essentials of a complicated new field, to select important problems which are accessible, and to apply intense energy and insight to their solution, making liberal use of deep results of other mathematicians in widely differing areas. The breadth of his mathematical knowledge is clearly visible to those who know him and his work. He is also a fine writer of mathematics, and his lectures at Pisa and elsewhere are recognized for clarity which increases with the subtlety of the mathematical idea being explained.

Bombieri's work in number theory is related to many different topics, among them the geometry of numbers, elementary proofs of the prime number theorem, character sums in finite fields, zeros of L-functions, sieve methods, and the distribution of prime numbers. In the following remarks we confine our attention to the latter two of these subjects, to which Bombieri has made striking contributions.

In 1944 the Russian Academician Ju. V. Linnik invented the large sieve, as a statement concerning the distribution of an arbitrary finite set of integers in arithmetic progressions. This sieve is formulated as an inequality which asserts roughly that the set in question must be reasonably well distributed in arithmetic progressions, unless the set is sparse. Following important works of A. Rényi and K. F. Roth, Bombieri played a major role in sharpening the large sieve, and he found many important applications of the large sieve to problems in number theory.

In 1965 Bombieri used his sharpened form of the large sieve to obtain a new and basic result concerning the distribution of prime numbers in arithmetic progressions. His result, now widely known as Bombieri's mean value theorem, may be described as follows. Let $\pi(x;\, q,\, a)$ denote the number of primes $p \leqslant x$ such that p is in the arithmetic progression $a,\ a+q,\ a+2q,\ a+3q.\ .\ .$ If the integers a and q have a common factor larger than 1 then there can be at most one such prime. If, on the other hand, the greatest common divisor of a and q is 1, then Dirichlet's classical theorem asserts that there are infinitely many such primes. The prime number theorem for arithmetic progressions asserts more, namely that $\pi(x;\, q,\, a)$ is approximately $\pi(x)/\phi(q)$, where $\pi(x)$ is the total number of primes $p \leqslant x$, and $\phi(q)$ is the number of integers a, $1 \leqslant a \leqslant q$, such that $(a,\, q) = 1$. Thus the prime numbers are approximately evenly divided into the arithmetic progressions in which they can lie. However, this result is known to hold only for rather small values of q, say $q \leqslant (\log x)^A$. In many questions it is useful to know about the size of the error

$$E(x;\, q,\, a) = \pi(x;\, q,\, a) - \pi(x)/\phi(q)$$

for rather large values of q. Bombieri's mean value theorem asserts that $E(x;\, q,\, a)$ is small on average

$$\sum_{q \leq Q} \max_{\substack{a \\ (a,q)=1}} |E(x;\, q,\, a)| < Cx(\log x)^{-A} \qquad (1)$$

provided that $Q < x^{1/2}(\log x)^{-B}$; here A can be taken to be arbitrarily large, and $B = B(A)$.

Estimate 1 is very useful in investigations concerning prime numbers. For example, in 1961 Linnik gave a long and complicated proof that every large positive integer n is a sum of a prime and two squares ($n = p + x^2 + y^2$, x, y integral). Earlier C. Hooley had proved this result in a more straightforward manner, but subject to an unproved hypothesis; estimate 1 can be used in place of this hypothesis to make Hooley's argument unconditional. Although the twin prime hypothesis (that there are infinitely many pairs of primes p, p' with $p - p' = 2$) and the related Goldbach conjecture (that every even integer $n > 2$ is a sum of two primes, $n = p + p'$) remain unproved, estimate 1 permits us to make closer approaches to these conjectures, in a variety of ways. The twin prime hypothesis may be thought of as a question of the minimum distance between consecutive primes. The prime number theroem tells us that the average size of $p_{n+1} - p_n$ is $\log p_n$. Accordingly, if we put

$$E = \limsup_{n \to \infty} (p_{n+1} - p_n)/\log p_n$$

then clearly $E \leqslant 1$; of course the twin prime hypothesis implies that $E = 0$. Before 1966 it was known that $E \leqslant 15/16$; this had been proved by G. Ricci, in Milan, who was Bombieri's original mentor. After studying under Ricci, Bombieri came under the influence of H. Davenport, at Trinity College, Cambridge, and it was in 1966 that Bombieri and Davenport showed that

$$E \leqslant \frac{1}{8}\,(2 + 3^{1/2}) = 0.46650\ldots$$

The germ of their proof can be traced back to an unpublished manuscript "Some problems of 'Partitio Numerorum' (VII)" of G. H. Hardy and J. E. Littlewood, but it is only by repeated applications of estimate 1 that technical difficulties can be overcome to achieve a good result.

Estimate 1 is also useful in efforts to establish Goldbach's conjecture, and some progress has recently been made toward that end. In another recent advance, this spring at the Institute for Advanced Study in Princeton, New Jersey, Bombieri invented a new asymptotic sieve which raises the possibility of coming still closer to Goldbach's conjecture.

Bombieri's introduction to minimal surfaces, the second area of his work cited in the Fields Medal award, was due to E. De Giorgi at Scuola Normale Superiore, a few blocks away from the University of Pisa where in 1966 Bombieri assumed his present chair. For some years De Giorgi and his school in Pisa had been studying and making major contributions to problems in the area of mathematics now generally called "geometric measure theory" (the title of the treatise by H. Federer on the subject).

An initial mathematical difficulty in studying multidimensional surfaces in general arises when they are allowed to have irregular or singular points (cusps, edges, corners, or much more complicated geometric configurations). Indeed the presence of such singular sets seems to be the rule in many applications; for example, spider webs, crystal shapes, and compound soap bubbles are all minimal surfaces of special types. Another rich source of surfaces with singularities is the general problem of finding surfaces of least area spanning a prescribed boundary.

Because of the central role of area in defining surfaces, and also for other historical and especially mathematical reasons, the problem of finding surfaces of least area (frequently called "area minimizing" or "minimal" surfaces) has become a central one in the modern calculus of variations. (This problem is often called "Plateau's problem" in honor of the Belgian physicist of the last century who studied, among other things, the geometry of soap films.) One of Bombieri's contributions to this theory lies in helping to show the existence of singular points of area-minimizing surfaces in an especially nice case. Before this work the possibility was open that if one considered only surfaces of least area which locally were the boundaries of sets (this is the main case studied by De Giorgi's school —in particular, a 13-dimensional minimal surface has to lie in a 14-dimensional space) there might be no singularities at all. Indeed, the work of De Giorgi, E. R. Reifenberg, W. H. Fleming, F. J. Almgren, Jr., and J. Simons implied there were no interior singular points of such n-dimensional minimal surfaces in R^{n+1} (the euclidean space of $n + 1$ dimensions) for $n = 1, 2, 3, 4, 5, 6$. Bombieri, De Giorgi, and E. Giusti, however, produced a 7-dimensional surface of least area in R^8 which contained an essential singular point (this was supposed to be the result of one long night's work in Pisa); it then follows that such singularities exist for all $n = 7, 8, 9, \ldots$. In particular, the character of this formulation of Plateau's problem changes at dimension 7.

Another result of that night's work was the proof of the existence of a function f: $R^8 \rightarrow R$ whose graph in R^9 is a minimal surface which is not a hyperplane. If f: $R^n \rightarrow R$ has a graph which is a minimal surface, then S. Bernstein, Fleming, De Giorgi, Almgren, and Simons, among others, had

Enrico Bombieri [Courtesy University of British Columbia]

proved that this graph is a hyperplane provided n is no larger than 7. The significance of this counterexample to further extensions of "Bernstein's theorem" is that it provided a specific example of a class of nonlinear elliptic partial differential equations (the f's satisfy the so-called minimal surface equation), the solution sets of which change dramatically with increasing dimension—for nonlinear vector-valued elliptic partial differential equations the possible changes of solutions with dimension are now known to be even more dramatic.

There being numerous globally defined nontrivial solutions to the minimal surface equation in higher dimensions, it becomes a problem of some interest (both intrinsically and also because of possible applications to solutions of other nonlinear elliptic equations) to understand the structure of the space of solutions in a fixed dimension. Such a program seems to require analytic estimates both for the functions f and for functions u defined on the minimal surface itself and satisfying partial differential equations defined on the surface. Bombieri, in collaboration with De Giorgi, Giusti, and M. Miranda, has made contributions in both these directions. In particular, seemingly optimal estimates have been obtained for the size of the gradients of such functions f in terms of the oscillation of f in a neighborhood, and a Harnack-type inequality has been obtained for such functions u.

Bombieri has traveled widely in the United States as well as in Europe, and the diversity of his mathematics is in part a reflection of this. While he was visiting Columbia University in 1969–1970, S. Lang attended his lectures on minimal surfaces and realized that analytic estimates of the type proved in the study of minimal surfaces were similar to the estimates required to settle questions in transcendental number theory. Lang and Bombieri wrote a joint paper in this direction, and a short time later Bombieri incorporated an L_2 estimate of L. Hörmander on plurisubharmonic functions to solve a problem related to the algebraic values of meromorphic maps.

Like a number of other mathematicians, Bombieri became interested in mathematics at a fairly early age. At 13, for example, he was studying a textbook in number theory. At the present time he has well-developed tastes in both food and wine (some of the latter from his family's vineyard). He is also an avid player of chess, bridge, and poker, and frequently adds to his impressive collections of stamps and seashells. The story is told that his father has promised him a Ferrari when he settles the Riemann hypothesis.

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