vidual homes (7), researchers are studying, among other materials, organic materials, such as paraffin waxes, for storage where a long cycle life is required. For the higher temperatures needed in solar energy central power generating plants, several laboratories are studying eutectic salts, such as sodium nitrate–sodium chloride.

A number of other storage options are also emerging, and observers say there will likely be several more. For example, steam generated in central power plant boilers might be stored in tanks or caverns until it is needed to run turbines. Coal gasifiers, when operational, are expected to be most efficient when operated continuously. The gas produced might similarly be stored at high pressure in caverns. A rate structure for energy that reflected the cost of energy (base load electricity is cheaper to make than peak electricity) might reshuffle consumption patterns so that the demand for energy was more uniform. Or a new storage scheme may enter the scene with the potential of becoming dominant in its field. For example, recently, J. A. Van Vechten of the Bell Laboratories, Murray Hill, New Jersey, described a thermal storage scheme where molten semiconductors were used as a heat-of-fusion storage material (8). Van Vechten suggested that substantial improvements in storage efficiencies (because of larger heats-of-fusion and better heat transfer) and lower costs could result from the use of semiconductors.

Despite the importance of energy storage, the activity as indicated by funding can only be described as modest. The largest advanced storage research program in the federal government belongs to the Atomic Energy Commission, which is spending about $1.8 million this year, the largest part going toward batteries. Overall, probably less than $10 million is being spent throughout the United States on advanced energy storage (that is, excluding pumped hydroelectric and commercial batteries). A possible consequence is that some time will pass before any of the advanced energy storage systems become available.

—ARTHUR L. ROBINSON

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The Finite Element Method: A Mathematical Revival

The ideas that underlie a means to approximate solutions to differential equations, the finite element method, were first proposed by a mathematician, the late R. Courant of New York University, in 1943, but received little attention at that time. Even when other mathematicians again proposed similar ideas in 1953, mathematicians did not develop this method of approximation. Instead, the finite element method was developed by engineers who found other approximation schemes inadequate to enable them to solve problems in structural mechanics and elasticity. The method proved highly successful when applied to these engineering problems. As it came into extensive use, its defects as well as its advantages became clear and further analysis of the ideas on which it is based became necessary. Now, however, mathematicians are studying these ideas again and, together with engineers, are applying the finite element method to some extremely difficult practical problems.

A differential equation can be solved numerically by techniques, such as finite difference methods, that are based on approximations to the derivatives of a function. Alternatively, it can be solved by techniques based on approximations to the function that satisfies the differential equation. The finite element method is a technique that allows for an approximation to such a function.

When applying the finite element method to the solution of a differential equation, analysts consider a variational form of the equation. A variational form is an expression that can be derived from certain differential equations. It consists of a sum of integrals. A function that minimizes a variational form is also a function that satisfies the associated differential equation. By enabling investigators to approximate a function that will minimize a variational form of a differential equation, the finite element method allows for an approximation to a solution to that equation.

An approximation to a function that minimizes a variational form is constructed from combinations of certain trial functions. These trial functions are defined on the region in which a solution to the differential equation is sought. The region is divided into a grid, and the divisions of the region are called elements. In a two-dimensional problem (such as the problem of describing the forces on a vibrating membrane) the elements are usually triangles or rectangles.

Each trial function is zero on all parts of the region except for one element. The various trial functions—which are simple linear functions, polynomials of low degree, or the like—are joined together at the boundaries of the grid elements. Values of the trial functions are defined at certain points (nodes) of the elements, and for a given trial function, sufficiently many nodal values are stipulated so that only one function could satisfy all of those values.

In order to use a combination of these trial functions to approximate the solution to an equation, analysts must find the combination that will minimize a variational form of an equation. They thus formulate a sum of trial functions to be substituted into a variational form, each term of the sum consisting of a trial function multiplied by a constant. The constants are undetermined when the sum is formulated. The goal is to select a specific combination of constants that will result in a minimization of a variational form when the sum is substituted into that expression. Such a collection of constants can be determined when a matrix equation is solved on a computer. The entries of the matrix correspond to the trial functions. The unknowns in the equation are the constants.

The approximate solution can be improved when the distance between nodes is decreased (that is, the grid is made finer) and a new collection of constants and thus a new approximate solution is calculated. In this way, a sequence of increasingly accurate approximations that converge to the exact solution can be obtained.

Matrix Equations

Since the entries of a finite element matrix are derived from the trial functions, the properties of the matrix reflect the structure of the trial functions. For example, since each trial function is zero over most of its domain, the matrices contain many zeros. These matrices are so constructed that their equations are numerically stable, a property essential to reliable computer calculations. (Small changes in the problem result in only small changes in the solution.) Numerical stability is essential to reliable computer calculations. For example, computer round-off errors result in inevitable changes in the matrix problem, and it is necessary that such changes do not result in a distorted solution.

Despite the fact that finite element matrices are fairly simple, the matrices are often large and the size of the matrix increases as the distance between nodes is decreased (hence the approximation to the solution is improved). The computer operations necessary to solve most problems can be difficult and time-consuming. Thus a means to reduce the number of arithmetic operations needed to solve finite element matrix equations is of considerable importance.

A. George of the University of Waterloo in Canada recently showed that it is possible to use the properties of matrices that arise from the finite element method to reduce by an order of magnitude the number of arithmetic operations necessary to solve the matrix equations. George used a graphical model of a finite element matrix to develop a scheme for reordering the variables in the equation. He showed that the matrix equation in which the variables were reordered could be solved with fewer operations than could the original equation and that the amount of computer storage space used in the solution to these equations can, in theory, be reduced. Working with M. Schultz and S. Eisenstat of Yale University and F. Dorr of Los Alamos National Laboratory, George has now devised a computer storage scheme that allows him to solve finite element matrix equations with the calculated minimum amounts of computer operations and storage space. Thus his result is of practical as well as theoretical importance.

According to G. Fix of the University of Michigan, problems of efficient data management are often more important to finite element computations than are problems of reducing the number of arithmetic operations. In finite element computer schemes, it is often the case that only a small amount of the computer time is used for numerical operations, the remainder being used for data management. This situation arises because the two major finite element computer operations—the construction of trial functions and the elimination of unknowns in the matrix equation—are normally performed separately in finite element computer codes. Thus a method, developed by B. Irons of the University of Wales, in which these two operations are performed concurrently, is a significant contribution to this problem of reducing the amount of computer time expended in data management. Irons' method, which is commonly called the frontal method, was proposed only a few years ago but has already been put to extensive use.

Problems Are Altered

Despite the results of George and Irons, the process of solving finite element equations can be long and complicated. In addition, the incorporation of certain aspects of problems into the construction of trial functions may greatly increase the computational difficulty of the problems. In some cases, investigators have found that they could so alter a difficult problem as to make it easier to solve, and could obtain a solution to the altered problem that converges to the solution to the original problem.

One situation in which problems can be altered and their solutions facilitated occurs when certain problems are defined on irregularly shaped regions. The computational cost of extending trial functions to such a region is great. Investigators may then choose to ignore the irregular boundary and assume that the irregular region is a polygon. This change in the problem will result in a change in the approximate solution. The approximate solution to the altered problem will often converge more slowly to the true solution to the original problem. However, in many cases, the penalty associated with the alteration of the problem costs less in computer time and computational complexity than does the extension of the problem to the irregular boundary.

Computational problems that arise when the trial functions are extended to certain boundaries can also be circumvented by a method studied by I. Babuska of the University of Maryland. Babuska analyzed a variational form of a problem in which the solution is required to take on specific values at the boundaries of a region. The incorporation of these boundary values into the approximate solution to the problem is often technically complicated. Babuska avoided this problem by considering the sum of the original variational form of the problem and a weighted contribution of the boundary regions to the solution. He then minimized this sum by taking combinations of trial functions that have no boundary restrictions.

Babuska's method allows for an approximation that converges to the solution to the problem. The penalty associated with this technique is that convergence to the solution may be slower if the method is applied than if it is not applied. Babuska postulated, however, that it is theoretically possible to use his technique and have an arbitrarily small convergence penalty.

T. King of the University of Cincinnati has recently shown that an optimum convergence rate (no convergence penalty) can be achieved in approximations to the solutions of a class of boundary value problems. He used Babuska's method and computed the approximate solution to a given problem several times, using different weights for the boundary contribution in each calculation. By taking a linear combination of the solutions that result from these different boundary weights, he obtained a solution that converges at the theoretical maximum rate.

Another way to simplify problems involves the use of nonconforming trial functions. These do not obey certain physical restrictions on the problem, and, in many cases, are not continuous across the grid into which the region is divided. When such trial functions are defined, the variational form of the problem (which often represents an energy that is to be minimized) takes on infinite values at the boundaries of the grid. That the variational form must remain finite is a physical restriction on the problem. In order to obey this restriction when nonconforming trial functions are used, analysts calculate the approximate solution on each grid element separately. They then add up all of these approximations and ignore the grid boundaries.

Engineers discovered experimentally that certain changes in problems can result in only very small errors in the solutions. Only recently, however, have mathematicians analyzed how and when these problems may be changed. In certain cases, engineers have devised tests that allow them to determine whether a given problem can be altered. The first successful test, the patch test, was proposed by Irons in 1965.

Irons was studying problems in plate bending at the Rolls-Royce Company when he devised his test. He conjectured that the use of nonconforming trial functions is justified if his test criterion is satisfied. The test consists of an examination of a special case of convergence of the nonconforming trial functions. A small patch of grid elements is arbitrarily chosen, and the boundary conditions of the problem are arranged so that the true solution to the problem in that region is a polynomial. If the solution obtained by nonconforming trial functions defined on the elements of the patch is that polynomial, then the patch test is passed. The patch test was experimentally successful, but only recently (in 1972) was it mathematically justified by G. Strang of the Massachusetts Institute of Technology.

Alterations in problems such as those tested by Irons allow investigators to circumvent restrictions of the finite element method. Thus the analyses of these techniques are of considerable interest to engineers and mathematicians. Strang, Babuska, and Fix are among those mathematicians who are studying changes in problems to determine when they are and are not justified.

Superconvergence

Not all mathematically interesting aspects of the finite element method are of immediate practical importance. The phenomenon of superconvergence is of considerable interest to mathematicians but has not yet yielded results that have been applied to engineering problems. Superconvergence occurs when an approximate solution to a problem is much closer to the actual solution at some node points than would ordinarily be expected.

A recent description of superconvergence involves the finite element approximation to the heat equation, an equation that describes the conduction of heat through an object as a function of time. J. Douglas and T. Dupont of the University of Chicago report that superconvergence occurs at node points when all of the finite element trial functions are polynomials of at least the second degree and the trial functions are pieced together so that they are continuous but not necessarily differentiable across the grid boundaries. Douglas and Dupont have expressed this convergence in terms of the polynomials used in the approximation. Superconvergence is more pronounced as the degree of the polynomials increases. Superconvergence of the heat equation has not yet been applied to engineering problems, but Dupont believes that practical applications of such results cannot be ruled out.

According to Douglas, there is another type of mathematical research of superconvergence that is likely to be of practical importance. Such research, which is done by J. Bramble and A. Schatz of Cornell University, involves auxiliary calculations with the approximate solution obtained by the finite element method. Such calculations may be useful to engineers when they are less interested in an approximate solution than in quantities calculated from it. For example, in certain problems studied by Douglas and Dupont together with M. Wheeler of Rice University in Houston, flow rates are obtained from the first derivatives of the approximate solutions.

The type of superconvergence studied by Bramble and Schatz is a superconvergence in regions rather than points of the approximate solution. Their result follows when they take local averages of the approximate solution about arbitrary points. This technique may facilitate the application of superconvergence to studies of properties such as flow rates that are calculated from the approximate solution in a region of the domain.

The recent interest among mathematicians in the finite element method has coincided with successful applications of the technique to some extremely difficult practical problems. In the past decade the method was more often applied to problems in solid mechanics since variational forms of such problems are easily obtained. Variational forms of problems in fluid mechanics are more difficult to construct. Thus problems in fluid mechanics have only recently been approached by the finite element method.

Many problems that arise in studies of blood circulation can be described by equations of fluid mechanics. P. Tong of the Massachusetts Institute of Technology together with Y. Fung of the University of California at San Diego have applied the finite element method to equations that describe the flow of viscous fluids. Their results, they say, can be used to develop a numerical method of analysis of problems of describing blood circulation.

The finite element method is also being used to study the dynamics of ocean circulation. Fix is developing models that describe the time-dependent and highly nonlinear meandering of the Gulf Stream. This process is believed to be important to the global pattern of ocean circulation. Another phenomenon that Fix hopes to model with the finite element method is the so-called midocean meso-scaled eddies. These eddies, whose flow is also time dependent and highly nonlinear, transport a significant amount of energy.

Another difficult problem in fluid mechanics is that of finding approximate solutions to neutron diffusion and transport equations. Solutions to these equations are important to the design of nuclear reactor cores. Hans Kaper and his colleagues at Argonne National Laboratories are now applying the finite element method to these equations. According to Kaper, the finite element method may lead to a general solution method for these equations since it already has been used to solve particular classes of such problems.

Recent applications of the finite element method to increasingly complex and varied problems are possible because communications between engineers and mathematicians have increased. Both engineers and mathematicians are enthusiastic about the upsurge of interest in the finite element method since they believe that current mathematical research will provide a theoretical basis for the future development of this highly successful approximation technique.—GINA BARI KOLATA

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