—much like the phase diagram for a gas-liquid system. Such a phase diagram can be constructed from luminescence measurements at different temperatures (Fig. 1).

Calculation of any part of the phase diagram at finite temperatures is difficult, because the theory is tractable only at $T = 0°K$. By ignoring the details of germanium's band structure and omitting treatment of the correlation energy, R. N. Silver of the California Institute of Technology was able to derive the equivalent of a van der Waals equation of state for the electron-hole system, from which such a phase diagram may be calculated.

As in all gas-liquid systems, there should be a critical value of the temperature and the density at which the exciton phase and droplet phase become indistinguishable. This critical point is of some interest because scientists are unsure of the nature of the fluid there. For example, some believe that, near the critical point, the excitons are largely dissociated into free electrons and holes and the droplet has lost its distinctly metallic character. Thus, near the critical point, both the gas and the liquid phases are like a classical (not high density) electron-hole plasma.

So far, no published data exist for germanium concerning the critical point. A rough approximation has enabled Combescot to extend the binding energy calculation to finite temperatures, and she estimated the critical temperature to be about 8°K and the density of electron-hole pairs at the critical point to be about 7×10^{16} cm^{-3}.

All of the experiments reported above have used fluxes of photons or electrons to produce high densities of electrons and holes. However, R. B. Hammond, T. F. Lee, V. Marello, J. W. Mayer, T. C. McGill, and Silver at Caltech have recently demonstrated that droplets can be produced in both silicon and germanium by direct injection of electrons and holes through electrical contacts on the ends of a sample. This method may prove advantageous in measuring the properties of electron-hole droplets under somewhat different conditions from those of previous experiments. Such electrical injection would also be preferable if the condensation of electron-hole droplets should ever become important in making useful opto-electronic devices.

Still to be answered are a number of questions concerning the properties of droplet surfaces, the nature of the nucleation process when droplets condense, and possible practical applications. As work continues on the nature of electron-hole droplets, it will doubtless become clear how much new knowledge of lasting significance emerges from the study of this phase. So far, the observation of a well-known process (gas-liquid transition) in a totally new environment (multicomponent quantum plasma) has afforded the solid state theorist an opportunity to test his hypotheses of electron behavior in a unique form of matter.

—ARTHUR L. ROBINSON

Combinatorics: Steps toward a Unified Theory

The classical mathematical treatment of continuous phenomena has proved indispensable to the other sciences. However, a different type of mathematics is required to describe the discontinuous processes that occur in such areas as network analysis, molecular evolution, and computer science. These processes can best be described with combinatorics, the mathematics of discrete phenomena.

According to Gian-Carlo Rota of the Massachusetts Institute of Technology, combinatorics is one of the least developed fields of mathematics. He attributes this lack of development to the extreme difficulty of combinatorial problems and to the lack of a unified theory that would give direction to those who attempt to solve such problems. Now some mathematicians are optimistic that a unified approach to combinatorics may be forthcoming. Recently, two outstanding problems have been solved and techniques for solving them are being generalized. Research on a third major problem is also leading to new methods that appear to have extensive applications.

The proof by Richard Wilson of Ohio State University of an important conjecture that involves the existence of various block designs has aroused considerable interest among combinatorial mathematicians. Block designs are arrangements of a set of distinct objects (points) into a collection of subsets (blocks). The arrangement is restricted by the requirement that various groups of points occur specific numbers of times in the blocks. Of considerable practical importance in the design of statistical experiments, block designs are used to reduce the number of experiments necessary to prove a hypothesis.

One class of block designs consists of Latin squares—a type of design introduced by Euler who, in 1782, made a famous conjecture about such designs. Euler was asked to arrange 36 army officers from 6 ranks and 6 regiments in a square array so that each rank and each regiment was represented in each row and each column of the array. This led to the more general problem of deciding when such arrangements are possible. Euler convinced himself that it is impossible to arrange the 36 officers in the desired array and then conjectured that no such arrangement exists for any array of n^2 objects when n minus 2 is divisible by 4. This conjecture was finally shown to be incorrect by R. C. Bose of Colorado State University in Fort Collins, S. S. Shrikhande of Bombay University in India, and E. T. Parker of the University of Illinois. While resolving Euler's problem, these mathematicians developed two new research techniques (which were also independently developed by H. Hanani at the University of Nagev in Israel). Wilson then used similar techniques in his proof of a conjecture that certain designs can exist if the designs are permitted to consist of sufficiently many points.

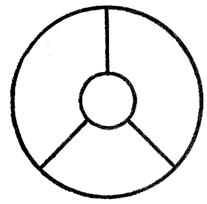
The first technique of Bose, Shrikhande, and Parker, and Hanani is recursive and allows the construction of large block designs from smaller designs. The second technique is applied to the direct construction of designs from finite algebraic structures which are used to specify the points of the blocks. According to Wilson, these two techniques provide a systematic method to prove the existence of various block designs.

Coloring problems are another class of combinatorial problems which, like block designs, are now being approached with unified theories. One coloring problem is that of assigning various "colors" or symbols to regions of a surface. A general statement about colorings of maps, the Heawood conjecture, has been solved for nonplanar surfaces by Gerhard Ringel and the late J. W. T. Youngs of the University of California in Santa Cruz, together with Charles Terry and Lloyd Welsh of the Institute for Defense Analyses in Princeton, New Jersey. The techniques are now being reformulated by Jonathan Gross of Columbia University and his colleagues Seth Alpert of the Downstate Medical Center in New York and Thomas Tucker of Colgate University. They are applying them to other problems, including some in other fields of mathematics.

The Heawood conjecture concerns colorings of maps, which are defined as divisions of a surface into nonoverlapping regions. It specifies the number of colors necessary to color any map on any surface so that no two adjacent regions of the map are assigned the same color. For example, according to the Heawood conjecture, any map on a torus (a doughnut-shaped object) can be colored with at most seven colors. Different numbers of colors are required to color maps on other surfaces.

The proof by Ringel, Youngs, and Terry of the Heawood conjecture for nonplanar surfaces, according to Terry, appeared to be uniquely applicable to that problem. They transformed maps into mathematically equivalent graphs and used graph theory to solve the problem. However, Gross noticed that this proof could be expressed in terms of covering spaces of algebraic topology. Covering spaces enable mathematicians to project a given region of one surface onto another surface where the region may be reproduced many times. Gross believes that his reformulation of the proof of the Heawood conjecture for nonplanar surfaces will lead to applications of that proof to problems in group theory, number theory, and topology. Moreover, Alpert has noticed a connection between this work with covering spaces and triple systems (a form of block design in which each block contains three points). Gross believes that this connection will facilitate research on such designs.

The one case of the Heawood conjecture that has not been proved—the Heawood conjecture for planar surfaces—is one of the oldest and most difficult problems in combinatorics. This case is usually called the four-color conjecture because it is a proposal that any map on the plane can be colored with at most four colors so that no two adjacent regions will be assigned the same color. The four-color conjecture is now being studied by a quantitative approach that, although it has not yet provided a solution to the problem, has led to useful research techniques.



A planar map that cannot be colored with fewer than four colors so that adjacent regions are assigned different colors.

The quantitative approach to the four-color conjecture involves the study of the behavior of chromatic polynomials. The chromatic polynomial $P(M,\lambda)$ associated with a map M gives the number of ways to color M with at most λ colors so that no two adjacent regions of M are assigned the same color. The four-color conjecture thus translates into the statement that, for any planar map M, $P(M,4)$ is greater than zero.

William T. Tutte and his associates at the University of Waterloo in Canada hope to develop the theory of chromatic polynomials to the point where they can use the classical theory of functions to obtain information about the class of functions $P(M,\lambda)$ evaluated at the point $\lambda = 4$. Accordingly, they are investigating the behavior of the class of functions $P(M,\lambda)$, when λ takes on values that are not integers.

Polynomials Are Evaluated

Evaluating the polynomials on a computer, Tutte and his colleagues have found that polynomials associated with several different maps have roots [values of λ for which $P(M,\lambda) = 0$] at values of λ between 3 and 4. Those roots are points of the Beraha sequence—a sequence of numbers proposed by Sami Beraha of Queens College in New York. Tutte is interested in looking for roots of chromatic polynomials at points of the Beraha sequence because he hopes to find a pattern in the behavior of chromatic polynomials that will enable him to infer whether or not the point $\lambda = 4$ is a root of some polynomial. The terms of the Beraha sequence come arbitrarily close to 4; hence, Tutte believes, further study of the Beraha sequence may lead to the determination of whether 4 is itself a root of $P(M,\lambda)$ for some map M or whether roots of various polynomials $P(M,\lambda)$ come arbitrarily close to, but never equal, the number 4. Such would be the case if the roots of the class of functions $P(M,\lambda)$ were only the points of the Beraha sequence.

In addition to their application to the four-color conjecture, chromatic polynomials may be applied to network analysis. Many problems in network analysis involve assignments of various "colors" or values to vertices of a graph. For example, Ronald Read of the University of Waterloo points out that chromatic polynomials can be used in network analysis to assign channels to television stations and to schedule classes in a university.

Chromatic polynomials can be used in network analysis because the problem of determining the number of ways to color the vertices of a graph is equivalent to the problem of determining the number of ways to color the regions of a map. A map containing n regions can be identified with a graph containing n vertices. Each region of the map can be associated with exactly one vertex of the graph. Two vertices of the graph are then connected if the two corresponding regions of the map have a common boundary. Thus chromatic polynomials can be incorporated into graph theory, an area of combinatorics that is useful to many other sciences such as engineering, chemistry, and statistical physics.

Unlike the recent research in block designs and coloring problems, research in some of the other areas of combinatorics has not yet led to unifying theories. For example, despite intense interest among computer scientists in computer algorithms, most problems in this area have been approached on an individual basis.

A computer algorithm is a sequence of logical instructions (steps) that enable a computer to solve a problem. Algorithms often consist of a large number of steps, and so it is desirable to analyze various algorithms to determine the minimum number of necessary steps. Researchers generally use special aspects of problems and of various computers when they attempt to minimize the number of steps of algorithms. Thus a solution to a given problem usually cannot be generalized to other problems.

As an example of the successful exploitation of unique aspects of a problem, L. H. Harper of the University of California at Riverside describes research on an algorithm used in studies of molecular evolution. Investigators of molecular evolution often use an algorithm that allows them to compare the nucleotide sequences of two DNA molecules and to find the longest or-

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dered subsequence of DNA that two DNA molecules have in common. If the two DNA molecules consist of m nucleotides and n nucleotides, respectively, an algorithm can find their longest common ordered subsequence in m times n steps. However, Peter Wiener of the Rand Corporation in Santa Monica, California, has now found an algorithm that can perform this operation in $m + n$ steps, a considerable saving if m and n are large.

Wiener made two assumptions that enabled him to find an algorithm consisting of $m + n$ steps. Neither assumption was included in the algorithm that requires mn steps. First, Wiener assumed that the algorithm must find only contiguous subsequences of DNA. (Because the genetic code is translated from contiguous sequences, it is not acceptable to choose a subsequence consisting of DNA nucleotides that are separated from each other on the DNA strand.) Second, Wiener assumed that the algorithm would be run on a random access computer—a computer with a large core memory that enables it to quickly recall information that is asked for by name. Computers with auxiliary memories, such as tape or drum memories, recall information much more slowly. Wiener's solution is thus extremely helpful to those who need this special case of the algorithm for research on molecular evolution and who have random access computers. It does not aid those who need the more general algorithm or who have more restricted computers. Although research on special cases usually precedes the development of unifying theories, unifying theories have not as yet been proposed for problems involving computer algorithms.

Ronald Graham of Bell Laboratories in Murray Hill, New Jersey, believes that the lack of combinatorial theories is due in part to the fact that combinatorics has traditionally not been considered a subject unto itself. Only recently, he says, have researchers emphasized the unity of combinatorics. Moreover, the large number of as yet unsolved problems makes it difficult to build theories. However, the recent rapid development of combinatorics is encouraging to those who, like Rota, believe that combinatorial theories are necessary to the future of the mathematical, physical, and life sciences.

—GINA BARI KOLATA